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Variational Dissipative Mechanics on Lie Algebroids

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Abstract

We formulate a Herglotz-type variational principle on a Lie algebroid and derive the corresponding Euler–Lagrange–Herglotz equations for a Lagrangian depending on an additional scalar variable z. This provides a geometric framework for dissipative systems on Lie algebroids and recovers, as special cases, the classical Euler–Lagrange–Herglotz equations on tangent bundles, the Euler–Poincaré–Herglotz equations on a Lie algebra, and the Lagrange–Poincaré–Herglotz equations on Atiyah algebroids of principal bundles. Starting from the local formulation, we then use Lie algebroid connections to obtain a global connection-based Euler–Lagrange–Poincaré-Herglotz and Hamilton–Pontryagin–Herglotz theory, where the connection serves as an auxiliary device for the horizontal-vertical splitting of the dynamics. Finally, we establish energy balance laws and Noether–Herglotz-type results, in which classical conserved quantities are replaced by dissipated invariants.

Keyword Herglotz variational principle · Lie algebroids · dissipative systems · contact mechanics · Hamilton–Pontryagin principle

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1 Introduction

Variational principles lie at the heart of classical and modern mechanics. In the Lagrangian picture, the dynamics of a mechanical system with configuration space Q

This paper is dedicated to our friend Professor Juan Carlos Marrero on the occasion of his 60th birthday.

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and Lagrangian $L: TQ \to \mathbb{R}$ is obtained by requiring that the action functional

$$S[q] = \int_0^T L\big(q(t), \dot{q}(t)\big)\, dt$$

be stationary under variations of the curve $q(\cdot)$ with fixed endpoints. The associated Euler–Lagrange equations provide a classical description of conservative dynamics on TQ, and their reductions, under a symmetry group, yield the familiar Euler–Poincaré and Lagrange–Poincaré equations (Holm et al. 1998; Cendra et al. 2001; Bobenko and Suris 1999a, b).

For systems with dissipation, the standard variational framework is not directly applicable: dynamics under Rayleigh friction forces, thermodynamic couplings, or interaction with a reservoir typically is not described by critical values of the classical action functional (Gay-Balmaz and Yoshimura 2017). One way to circumvent this difficulty, going back to Herglotz, is to replace the action integral by an *action variable* $z(t) \in \mathbb{R}$ solving

$$\dot{z}(t) = L\big(q(t), \dot{q}(t), z(t)\big),$$

and to impose stationarity of the terminal value $z(T)$ under variations of $q(\cdot)$. The resulting Euler–Lagrange–Herglotz equations provide an alternative variational description of a class of dissipative systems and are closely related to contact geometry and contact Hamiltonian dynamics (De León and Valcázar 2019; de León and Valcázar 2020; Colombo et al. 2025a, b).

On the other hand, Lie algebroids offer a unifying language for geometric mechanics and reduction (Weinstein 1996). They encompass different phase spaces such as the tangent bundle TQ, Lie algebras, action algebroids and Atiyah algebroids of principal bundles. Thus, they form a natural stage for symmetry-reduced systems. Lagrangian and Hamiltonian mechanics on Lie algebroids have been developed in a series of works (Martínez 2001; de León et al. 2005; Grabowska et al. 2006), and the corresponding Euler–Lagrange–Poincaré and Hamilton–Pontryagin formalisms have been recently refined in the connection-based approach of Li et al. (2017) and of Hu and Stern (2024). In this framework, many reduced or constrained equations of motion arise from a single geometric principle on an abstract algebroid.

Recent years have seen a growing interaction between contact mechanics and geometric mechanics on Lie algebroids, as they provide a unified framework to study dissipative systems, despite the fact that the appropriate geometric framework for dissipative Lagrangian systems is still an active subject of research. In particular, while the Herglotz principle provides a natural variational route to a broad class of dissipative equations, it should not be understood as the unique possible Lagrangian formalism for dissipation. Alternative approaches exist, such as the Tulczyjew-type contact Lagrangian formalism developed in Grabowska and Grabowski (2024).

The aim of this paper is to develop a direct *Herglotz variational principle on a Lie algebroid* $(E \to M, [\cdot,\cdot], \rho)$, as a variational framework for dissipative reduced dynamics. We consider a Herglotz-type Lagrangian $L: E \times \mathbb{R} \to \mathbb{R}$, $(x, y, z) \mapsto L(x, y, z)$, and an additional scalar variable $z(t)$ evolving according to $\dot{z}(t) = L\big(x(t), y(t), z(t)\big)$, along an admissible curve $(x(t), y(t))$ in E. Requiring stationarity of $z(T)$ under admissible variations E-homotopic to (x, y) leads to the system of *Euler–Lagrange–Herglotz equations on the Lie algebroid*, derived in Simoes et al. (2024a). The variational principle and equations reduce to the classical Herglotz variational principle when $E = TQ$. In addition, in order to present the minimal amount of material for our purposes, we adopt a similar approach to variational calculus as that in Martínez (2008) and extend it to the contact setting, though there are equivalent more intrinsic approaches (see (Grabowska and Grabowski 2008) or the optimal-control approach on almost Lie algebroids (Grabowski and Jóźwikowski 2011)). Our viewpoint in this paper is more specific: we work within the Herglotz variational setting and develop a direct Lie-algebroid formulation, without relying on the full prolongation-based machinery (Simoes et al. 1923), in a form that is well suited to explicit reduced equations and especially to future discretization developments. In particular, the use of connections later in the paper should be understood only as an auxiliary device for obtaining a global rewriting of the dynamics, rather than as part of the geometric data of the variational problem.

Our first main result is a direct variational derivation of the Euler–Lagrange–Herglotz equations on $(E, [\cdot, \cdot], \rho)$ in local coordinates. Although some of the resulting equations recover known intrinsic constructions in specific settings, the point here is that they arise from a single Herglotz-type variational scheme on Lie algebroids, in a form adapted to explicit reduction procedures and later discretization. We then recall several familiar classes of equations that appear as special cases such as the classical Euler–Lagrange–Herglotz equations on TQ; the Hamel–Herglotz equations in quasi-velocities; the Euler–Poincaré–Herglotz equations on a Lie algebra and on an action Lie algebroid; and the Lagrange–Poincaré–Herglotz equations on the Atiyah algebroid of a principal bundle. In particular, we obtain a Herglotz-type, gauge-invariant dissipative extension of Wong's equations on an Atiyah algebroid, together with a thermomechanical interpretation for certain rigid body models with entropy-like variables.

Our second main contribution is to analyze the *Noether-type* properties of Herglotz systems on a Lie algebroid. It has already been proven in Simoes et al. (2024a, 1923) that the Lagrangian energy is dissipated by a factor depending on $\partial L/\partial z$. This behavior is a natural consequence of the fact that Herlotz equations are Hamiltonian vector fields with respect to a Jacobi structure, whose Hamiltonian function is the energy. For typical contact Lagrangians, the energy becomes a *dissipated quantity* in the sense of De León et al. (2021) and, after multiplication by a suitable integrating factor, it is conserved along solutions. An analogous result holds for Noether-type momenta associated with infinitesimal symmetries of L. We illustrate these ideas with examples including mechanical systems with Rayleigh dissipation, dissipative Wong equations, and rigid bodies coupled to entropy-like variables. Although, these dissipative properties are not surprising and mimic the behavior of Herglotz-type systems in the phase space $TQ \times \mathbb{R}$, its analysis has not been investigated in the previous literature.

The third main ingredient of the paper is a global connection-based reformulation of the Herglotz variational equations on a Lie algebroid, based on TM-connections and the Euler–Lagrange–Poincaré viewpoint of Li et al. (2017); Hu and Stern (2024). The connection is used as an auxiliary device to split the dynamics into horizontal and vertical components, and in Appendix A we make explicit that the resulting local equations are independent of the chosen connection. Given a TM-connection ∇ on E and the induced E-connection ∇^* on E^*, we show that the local Euler–Lagrange–Herglotz equations are equivalent to the intrinsic relation

$$\overline{\nabla}^*_{a(t)} p(t) - \rho^*\big(dL_{\mathrm{hor}}(a(t), z(t))\big) = \frac{\partial L}{\partial z}\big(a(t), z(t)\big)\, p(t), \qquad p = dL_{\mathrm{ver}}(a, z),$$

for the momentum $p \in E^*$ along an admissible curve $a(t) \in E$.

Finally, we extend the Hamilton–Pontryagin principle on Lie algebroids to a *Hamilton–Pontryagin–Herglotz* principle, where the Herglotz variable z satisfies a differential equation including both the Lagrangian and a Lagrange multiplier, dual variable p, responsible for ensuring the additional constraint that a trajectory must be an E-path. The resulting implicit Euler–Lagrange–Poincaré–Herglotz equations provide a convenient starting point for discretization and structure-preserving numerical schemes on Lie groupoids. Several nontrivial examples are presented to illustrate the scope of the theory.

Beyond the continuous theory developed here, our main motivation is to prepare the ground for contact-type variational integrators on Lie groupoids.

The paper is organized as follows. Section 2 recalls basic notions on Lie algebroids and admissible curves. In Sect. 3, we introduce the Herglotz variational principle on a Lie algebroid leading to the already known Euler–Lagrange–Herglotz equations. Section 4 is devoted to a Noether–Herglotz theorem leading to dissipated momentum maps whose behavior is similar to that of the energy. Finally, in Sects. 5 and 6, we develop the Euler–Lagrange–Poincaré–Herglotz equations in a connection-based, coordinate-free form and formulate the Hamilton–Pontryagin–Herglotz principle on a Lie algebroid, respectively. In Sect. 7, we include several classical and new examples of Herglotz reduction-type equations illustrating our results.

2 Preliminaries on Lie Algebroids

Let $\tau : E \to M$ be a vector bundle of rank r over a manifold M of dimension n. Denote by $\Gamma(E)$ the set of sections of this vector bundle, by $\mathfrak{X}(M)$ the set of vector fields on M, and by $C^\infty(M)$ the set of smooth functions on M. If γ is a curve on M, we will denote by $\Gamma_\gamma(E)$ the set of sections of $\Gamma(E)$ evaluated along the curve γ.

A *Lie algebroid* structure on E consists of a Lie bracket $[\cdot, \cdot]\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and a vector bundle map over the identity $\rho\colon E \to TM$, or $\rho\colon \Gamma(E) \to \mathfrak{X}(M)$, called the anchor, satisfying the following Leibniz rule

$$[\sigma_1, f\sigma_2] = f[\sigma_1, \sigma_2] + (\rho(\sigma_1) f)\, \sigma_2,$$

for all $\sigma_1, \sigma_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

Locally, let (x^i), $i = 1, \ldots, n$ be coordinates on M, and $\{e_\alpha\}$, $\alpha = 1, \ldots, r$ a local basis of sections of E. Then the anchor and the bracket satisfy

$$\rho(e_\alpha) = \rho^i_{\ \alpha}(x)\,\frac{\partial}{\partial x^i}, \qquad [e_\alpha, e_\beta] = C^\gamma_{\ \alpha\beta}(x)\,e_\gamma,$$

for some smooth functions $\rho^i_{\ \alpha}$ and $C^\gamma_{\ \alpha\beta}$ called the anchor components and the structure functions, respectively. The Jacobi identity and the Leibniz rule impose useful identities on these functions.

The local basis of sections $\{e_\alpha\}$ induces the fibred coordinates (x^i, y^α) on E adapted to τ. A curve $a\colon [0, T] \to E$, written in local coordinates as

$$a(t) = \big(x^i(t), y^\alpha(t)\big), \qquad a(t) = y^\alpha(t) e_\alpha\big|_{x(t)},$$

is called *admissible* or an E-*path* if its base curve $x(t) = \tau(a(t))$ satisfies

$$\dot{x}^i(t) = \rho^i_{\ \alpha}\big(x(t)\big)\,y^\alpha(t). \tag{1}$$

The space of admissible curves generalizes the space of velocities $(q(t), \dot{q}(t))$ in TQ.

Following the standard variational calculus on Lie algebroids (Martínez 2008), we describe variations of admissible curves via E-*homotopies*.

Definition 1 An E-*homotopy* is a smooth map

$$a\colon (-\varepsilon, \varepsilon) \times [0, T] \to E, \quad (s, t) \mapsto a_s(t),$$

such that for each s, the curve $t \mapsto a_s(t)$ is admissible, i.e., $\dot{x}^i_s(t) = \rho^i_{\ \alpha}(x_s(t))\,y^\alpha_s(t)$, where $a_s(t) = (x_s(t), y_s(t))$. We say that a_s is a variation of the reference curve a_0.

Define the variation field $\frac{\partial a}{\partial s}(0, t) \in T_{a_0(t)}E$ to be given in local coordinates by

$$\delta x^i(t) := \left.\frac{\partial x^i_s(t)}{\partial s}\right|_{s=0}, \qquad \delta y^\alpha(t) := \left.\frac{\partial y^\alpha_s(t)}{\partial s}\right|_{s=0}.$$

A convenient way to parameterize variations of the reference curve $a_0(t)$ is via a time-dependent section $\xi(t) \in \Gamma(E)$ along $x_0(t)$, written locally as $\xi(t) = \xi^\alpha(t)\,e_\alpha|_{x_0(t)}$. Following Martínez (2008), for an admissible E-homotopy the variation field satisfies

$$\delta x^i = \rho^i_{\ \alpha}(x)\,\xi^\alpha. \tag{2}$$

and

$$\delta y^\gamma = \dot{\xi}^\gamma + C^\gamma_{\ \alpha\beta}(x)\,y^\alpha\,\xi^\beta. \tag{3}$$

In the variational principle presented in the next section, we will consider variations with fixed endpoints of the base curve $x(t)$. Locally, these are the variations of the form above satisfying $\xi^\alpha(0) = \xi^\alpha(T) = 0$.

Remark 1 In Section 3 of Martínez (2008), the definition of admissible variations is given in an intrinsic, global form. An E-homotopy is described as a Lie algebroid morphism $\Phi = a\,dt + b\,ds : T(I \times J) \longrightarrow E$, subject to the usual admissibility conditions and boundary constraints. Given an admissible curve $a(t)$, the infinitesimal variation associated with a time-dependent section $\sigma(t) \in \Gamma(E)$ along the base curve $\gamma(t) = \tau(a(t))$ is defined through the canonical morphism

$$\Xi_a : \Gamma_\gamma(E) \longrightarrow \Gamma_a(TE), \qquad \Xi_a(\sigma)(t) = \rho_1\big(\chi_E(\sigma(t), a(t), \dot{\sigma}(t))\big),$$

where $\chi_E : \mathcal{T}^E E \to \mathcal{T}^E E$ denotes the canonical involution in the prolongation algebroid $\mathcal{T}^E E$ and ρ_1 is the anchor of the vector bundle $\mathcal{T}^E E \to E$. In local coordinates (x^i, y^α) this yields

$$\Xi_a(\sigma)(t) = \rho^i_\alpha(\gamma(t))\,\sigma^\alpha(t)\,\frac{\partial}{\partial x^i} + \Big(\dot{\sigma}^\alpha(t) + C^\alpha_{\beta\gamma}(\gamma(t))\,a^\beta(t)\,\sigma^\gamma(t)\Big)\frac{\partial}{\partial y^\alpha}. \tag{4}$$

This expression provides a global, coordinate-free characterization of admissible variation fields on a Lie algebroid.

The definition used in this paper corresponds to the local expression of the operator Ξ_a above. There, the variation is introduced directly through a time-dependent section $\xi(t)$ along $x(t)$, leading to

$$\delta x^i = \rho^i_\alpha(x)\,\xi^\alpha, \qquad \delta y^\gamma = \dot{\xi}^\gamma + C^\gamma_{\alpha\beta}(x)\,y^\alpha\,\xi^\beta.$$

Comparing with (4), we see that

$$\Xi_a(\xi) = \delta x^i\,\frac{\partial}{\partial x^i} + \delta y^\alpha\,\frac{\partial}{\partial y^\alpha},$$

so both definitions are equivalent in content. The formulation in Martínez's paper (Martínez 2008) is fully geometric and independent of coordinates, while the one used here is its local version, more convenient for explicit computations in the Herglotz variational setting. ◇

3 Herglotz Variational Principle on a Lie Algebroid

Throughout the text, let (x^i, y^α) be fibred coordinates on E associated to the projection τ and to a choice of a local basis of sections. Throughout the text the algebroid structure of $\tau : E \to Q$ will be naturally extended to an algebroid over $E \times \mathbb{R} \to Q$ through the projection onto the first component. The Lie algebroid projection, the anchor map and the bracket of sections and all local functions will be written using the same notation.

Let $L\colon E \times \mathbb{R} \to \mathbb{R}$ be a smooth function, which we call a *Herglotz-type Lagrangian*. We introduce an additional scalar variable $z(t) \in \mathbb{R}$ satisfying the Herglotz differential equation

$$\dot{z}(t) = L\big(x^i(t), y^\alpha(t), z(t)\big). \tag{5}$$

An admissible trajectory solving the Herglotz equation is a triple $t \mapsto \big(x^i(t), y^\alpha(t), z(t)\big) \in E \times \mathbb{R}$, such that:

$$\dot{x}^i(t) = \rho^i{}_\alpha(x^i(t))\, y^\alpha(t), \qquad \dot{z}(t) = L(x^i(t), y^\alpha(t), z(t)). \tag{6}$$

Definition 2 We say that a triple $(x^i(t), y^\alpha(t), z(t))$ is a *Herglotz extremal* on the Lie algebroid if, for $\varepsilon > 0$ and any E-homotopy $(x^i_s(t), y^\alpha_s(t), z_s(t))$ with $(s,t) \in (-\varepsilon, \varepsilon) \times [0, T]$ such that:

1. for each $s \in (-\varepsilon, \varepsilon)$, (x^i_s, y^α_s, z_s) satisfies (6);
2. $x^i_s(0) = x^i(0)$, $x^i_s(T) = x^i(T)$ for all $s \in (-\varepsilon, \varepsilon)$;
3. $z_s(0) = z(0)$ for all $s \in (-\varepsilon, \varepsilon)$;
4. the following stationarity condition is satisfied

$$\left.\frac{d}{ds}\right|_{s=0} z_s(T) = 0. \tag{7}$$

In other words, $z(T)$ plays the role of an action functional, and we require it to be stationary under admissible E-homotopies. Indeed, as we will see next, the stationarity principle (7) is the variational principle for Euler–Lagrange–Herglotz equations on Lie algebroids.

In the following theorem, we will show that a curve in $E \times \mathbb{R}$ being a solution of the Herglotz equations on Lie algebroid is a necessary and sufficient condition for it to be a Herglotz extremal. Below, we assume that the curve is contained in a local chart but we will treat the coordinate-independent (intrinsic) case in Sect. 5.

Theorem 1 *Let $(E \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid whose anchor components are given by $\rho^i{}_\alpha(x)$ and structure functions are $C^\gamma{}_{\alpha\beta}(x)$ with respect to local coordinates denoted by (x^i, y^α, z). Let $L\colon E \times \mathbb{R} \to \mathbb{R}$ be a Herglotz-type Lagrangian and consider a curve entirely contained in a coordinate chart on which it has the local expression $(x^i(t), y^\alpha(t), z(t))$ satisfying*

$$\dot{x}^i(t) = \rho^i{}_\alpha(x^i(t))\, y^\alpha(t), \qquad \dot{z}(t) = L(x^i(t), y^\alpha(t), z(t)). \tag{8}$$

Then $(x^i(t), y^\alpha(t), z(t))$ is a Herglotz extremal, i.e., satisfies (7) for all admissible variations with $x^i(0), x^i(T), z(0)$ fixed, if and only if it satisfies the Euler–Lagrange–Herglotz equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + C^\gamma{}_{\alpha\beta}(x)\, y^\beta \frac{\partial L}{\partial y^\gamma} - \rho^i{}_\alpha(x)\, \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial z}\, \frac{\partial L}{\partial y^\alpha} = 0. \tag{9}$$

Proof Let $(x^i_s(t), y^\alpha_s(t), z_s(t))$ be an admissible variation of $(x^i(t), y^\alpha(t), z(t))$, i.e., for each $s \in (-\varepsilon, \varepsilon)$,

$$\dot{x}^i_s(t) = \rho^i{}_\alpha(x_s(t))\, y^\alpha_s(t) \text{ and } \dot{z}_s(t) = L(x^i_s(t), y^\alpha_s(t), z_s(t)),$$

with $x^i_s(0) = x^i(0)$, $x^i_s(T) = x^i(T)$, $z_s(0) = z(0)$. Define the infinitesimal variations

$$\delta x^i = \partial_s x^i_s|_{s=0}, \;\; \delta y^\alpha = \partial_s y^\alpha_s|_{s=0}, \;\; \delta z = \partial_s z_s|_{s=0}.$$

Differentiating the Herglotz equation (5) with respect to the variable s, we obtain

$$\frac{d}{dt}(\delta z) = \frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial y^\alpha}\,\delta y^\alpha + \frac{\partial L}{\partial z}\,\delta z,$$

which is a linear ODE for the infinitesimal variation δz. Rearranging:

$$\frac{d}{dt}(\delta z) - \frac{\partial L}{\partial z}\,\delta z = \frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial y^\alpha}\,\delta y^\alpha.$$

Let

$$\lambda(t) := \exp\left(-\int_0^t \frac{\partial L}{\partial z}(\omega)\,d\omega\right).$$

Then

$$\frac{d}{dt}\big(\lambda\delta z\big) = \lambda\left[\frac{d}{dt}(\delta z) - \frac{\partial L}{\partial z}\,\delta z\right] = \lambda\left(\frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial y^\alpha}\,\delta y^\alpha\right).$$

Integrating both sides of the previous equation from 0 to T and using $\delta z(0) = 0$ (since $z(0)$ is fixed), we get

$$\lambda(T)\,\delta z(T) = \int_0^T \lambda(t)\left(\frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial y^\alpha}\,\delta y^\alpha\right)dt.$$

Since $\lambda(T) \neq 0$, the stationarity condition $\delta z(T) = 0$ is equivalent to

$$\int_0^T \lambda(t)\left(\frac{\partial L}{\partial x^i}\,\delta x^i + \frac{\partial L}{\partial y^\alpha}\,\delta y^\alpha\right)dt = 0. \tag{10}$$

Now, using (2) and (3), we express $(\delta x^i, \delta y^\alpha)$ in terms of a time-dependent section $\xi(t) = \xi^\alpha(t)\,e_\alpha|_{x(t)}$ along $x(t)$, via:

$$\delta x^i(t) = \rho^i_{\;\alpha}(x^i(t))\,\xi^\alpha(t), \qquad \delta y^\gamma(t) = \dot{\xi}^\gamma(t) + C^\gamma_{\;\alpha\beta}(x^i(t))\,y^\alpha(t)\,\xi^\beta(t).$$

Substituting into (10), we obtain

$$\int_0^T \lambda(t)\left[\frac{\partial L}{\partial x^i}\,\rho^i_{\;\alpha}\,\xi^\alpha + \frac{\partial L}{\partial y^\gamma}\left(\dot{\xi}^\gamma + C^\gamma_{\;\alpha\beta}y^\alpha\,\xi^\beta\right)\right]dt = 0.$$

where we drop the functions inputs to simplify the reading. Rewriting:

$$\int_0^T \left[\lambda\frac{\partial L}{\partial x^i}\,\rho^i_{\;\alpha}\,\xi^\alpha + \lambda\frac{\partial L}{\partial y^\gamma}C^\gamma_{\;\alpha\beta}y^\alpha\,\xi^\beta + \lambda\frac{\partial L}{\partial y^\gamma}\dot{\xi}^\gamma\right]dt = 0.$$

We integrate by parts the term with $\dot{\xi}^\gamma$:

$$\int_0^T \lambda \frac{\partial L}{\partial y^\gamma} \dot{\xi}^\gamma \, dt = \left[\lambda \frac{\partial L}{\partial y^\gamma} \xi^\gamma \right]_0^T - \int_0^T \frac{d}{dt}\Big(\lambda \frac{\partial L}{\partial y^\gamma} \Big) \xi^\gamma \, dt.$$

The boundary term vanishes because $\xi^\gamma(0) = \xi^\gamma(T) = 0$ (fixed endpoints). Hence the stationarity condition becomes

$$\int_0^T \left\{ \lambda \frac{\partial L}{\partial x^i} \rho^i{}_\alpha - \lambda \frac{\partial L}{\partial y^\gamma} C^\gamma{}_{\alpha\beta} y^\beta - \frac{d}{dt}\Big(\lambda \frac{\partial L}{\partial y^\alpha} \Big) \right\} \xi^\alpha \, dt = 0.$$

where the middle term changes sign due to the skew-symmetry of the structure functions, i.e., $C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}$. Since $\xi^\alpha(t)$ are arbitrary functions vanishing at $t = 0, T$, we deduce the pointwise condition

$$-\frac{d}{dt}\Big(\lambda \frac{\partial L}{\partial y^\alpha} \Big) + \lambda \frac{\partial L}{\partial x^i} \rho^i{}_\alpha - \lambda \frac{\partial L}{\partial y^\gamma} C^\gamma{}_{\alpha\beta} y^\beta = 0.$$

Expanding the derivative:

$$\frac{d}{dt}\Big(\lambda \frac{\partial L}{\partial y^\alpha} \Big) = \dot{\lambda} \frac{\partial L}{\partial y^\alpha} + \lambda \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right).$$

and using $\dot{\lambda} = -(\partial L/\partial z)\lambda$ by definition of λ, we obtain

$$\dot{\lambda} \frac{\partial L}{\partial y^\alpha} = -\lambda \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha}.$$

Hence,

$$-\left[-\lambda \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha} + \lambda \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) \right] + \lambda \frac{\partial L}{\partial x^i} \rho^i{}_\alpha - \lambda \frac{\partial L}{\partial y^\gamma} C^\gamma{}_{\alpha\beta} y^\beta = 0.$$

Simplifying and dividing by λ, which is a nowhere vanishing function, we conclude

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) + C^\gamma{}_{\alpha\beta}(x)\, y^\beta \frac{\partial L}{\partial y^\gamma} - \rho^i{}_\alpha(x) \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha} = 0.$$

These are the desired Euler–Lagrange–Herglotz equations (9).

The converse (that any solution of (8)–(9) is stationary) follows by reversing the above computations, showing that then the integral (10) vanishes for all admissible variations. $\square$

Remark 2 The variational principle above should be seen in parallel with the variational principle introduced in Grabowska and Grabowski (2008). An extension of these techniques to the contact case provides a geometric variational principle alternative to ours.

Remark 3 It is worth emphasizing that the Euler–Lagrange–Herglotz equations (9) admit a fully geometric interpretation in terms of the contact and Jacobi structures developed in Simoes et al. (2024a, 1923); Grabowska and Grabowski (2024). In this sense, Theorem 1 provides a variational derivation of the intrinsic Herglotz dynamics obtained in those works.

More precisely, in Simoes et al. (2024a) the authors construct a canonical Jacobi structure on $E^* \times \mathbb{R}$ and derive the contact Lagrangian equations pulling back the Hamiltonian equations to $E \times \mathbb{R}$.

Likewise, the contact Lie algebroid formalism in Simoes et al. (1923) derives the intrinsic Herglotz equations using the prolongation $\mathcal{T}^E(E \times \mathbb{R})$ and the algebroid differential $\mathrm{d} = d^{\mathcal{T}^E(E\times\mathbb{R})}$. In particular, the solutions of the Herglotz equations are admissible trajectories of a SODE section $\Gamma_L : E \times \mathbb{R} \to \mathcal{T}^E(E \times \mathbb{R})$ determined by the equations

$$\iota_{\Gamma_L} \mathrm{d}(\eta_L) = \mathrm{d}(E_L) - R_L(E_L)\,\eta_L, \qquad \iota_{\Gamma_L}\eta_L = -E_L,$$

where $\eta_L : E \times \mathbb{R} \to (\mathcal{T}^E(E \times \mathbb{R}))^*$ is the contact section associated with L, the function E_L is the corresponding energy, the section $R_L : E \times \mathbb{R} \to \mathcal{T}^E(E \times \mathbb{R})$ is the Reeb section uniquely determined by the conditions $\iota_{R_L} \mathrm{d}\eta_L = 0$ and $\iota_{R_L}\eta_L = 1$, and satisfying $R_L(E_L) = -\frac{\partial L}{\partial z}$.

Therefore, Theorem 1 complements the geometric constructions of Simoes et al. (2024a, 1923) by providing a coordinate variational formulation of the Herglotz dynamics.

It is also worth noting that an alternative geometric picture could be studied based on the contact-geometric framework of Grabowska and Grabowski (2024), where contact Lagrangian and Hamiltonian mechanics are formulated through a contact version of the Tulczyjew triple, using the canonical contact structures on the jet bundles J^1L of line bundles and the associated Legendre-type transformations. ◇

4 Energy Balance and Noether–Herglotz Theorems

In this section, we study how the presence of a Herglotz (or contact-type) term modifies the usual conservation laws associated with a Lagrangian system on a Lie algebroid. While in the conservative setting, energy and Noether momenta are constants of motion, here they satisfy first-order balance laws driven by the derivative $\partial L/\partial z$ of the Herglotz Lagrangian with respect to the contact variable. We show that, after multiplication by a canonical integrating factor, these quantities become time-independent *dissipated invariants* in the sense of Bravetti et al. (2017); Gaset et al. (2020); De León et al. (2021).

4.1 Energy Balance Law

Next, we recall the *Herglotz energy* associated with a Lagrangian $L\colon E\times\mathbb{R}\to\mathbb{R}$ and the energy balance law along solutions of the Euler–Lagrange–Herglotz equations. We interpret the resulting rescaled conserved quantity.

Given the coordinates (x^i, y^α, z) on $E\times\mathbb{R}$ adapted to a basis of sections of E, the *Herglotz energy* is the smooth function on $E\times\mathbb{R}$ locally defined by

$$E(x^i, y^\alpha, z) = \frac{\partial L}{\partial y^\alpha}(x^i, y^\alpha, z)\, y^\alpha - L(x^i, y^\alpha, z). \tag{11}$$

The previous expression is the local version of the energy function as defined on Simoes et al. (2024a, 1923).

It is well-known that in the contact setting the energy is not a conserved quantity but obeys a specific law, often interpreted as a dissipation law for typical contact Lagrangian functions. In the Lie algebroid setting, the situation is not different. In fact, it was shown in Simoes et al. (2024a) that the energy is the contact Hamiltonian function of the Euler–Lagrange–Herglotz equations with respect to a Jacobi structure on the Lie algebroid and thus its behavior in time is well-known.

Let $(x^i(t), y^\alpha(t), z(t))$ be a solution of the Herglotz equations (8)–(9). Then, the energy E satisfies

$$\dot{E} = \frac{\partial L}{\partial z}\, E. \tag{12}$$

It is also interesting to note that one can also derive the following conserved quantity

$$\frac{d}{dt}\left(e^{-\int_0^t \frac{\partial L}{\partial z}(\omega)\, d\omega}\, E(t)\right) = 0$$

along the trajectories. In the terminology of De León et al. (2021), the conserved quantity inside the parenthesis is a *dissipated invariant*.

4.2 Noether–Herglotz Theorem on a Lie Algebroid

In the following, we develop a Noether–type theorem for Herglotz systems on Lie algebroids and show that momentum maps on Lie algebroids have a similar behavior to that of the energy. Given an infinitesimal symmetry of L, we construct the corresponding *Herglotz momentum* and prove a Noether–Herglotz balance law. The momentum is no longer conserved but obeys an equation of the same form as the energy balance, and its rescaled version is constant. We then apply this result to several examples: a rigid body on a Lie algebra with exponentially damped body angular momentum, a class of dissipative Wong-type systems where both the kinetic energy and the momentum become dissipated invariants, and a thermoviscous rigid body formulated as an Euler–Poincaré–Herglotz system, in which the Herglotz variable acquires a thermodynamic interpretation as an entropy.

Next, we will define a symmetry generated by a section $\sigma\in\Gamma(E)$. Recall that given a section σ, with local expression $\sigma(x^i) = \sigma^\alpha(x^i)e_\alpha$, we may consider two special sections in the prolongation Lie algebroid $\mathcal{T}^E(E \times \mathbb{R}) \to E \times \mathbb{R}$: the vertical lift σ^V

$$\sigma^V(v_q, z) = \left(0_q, \left.\frac{d}{dt}\right|_{t=0} (v_q + t\sigma(q))\right),$$

whose local expression is

$$\sigma^V(x^i, y^\alpha, z) = \left(0_q, \sigma^\alpha(x^i)\frac{\partial}{\partial y^\alpha}\right),$$

and the complete lift σ^C whose local expression is

$$\sigma^C(x^i, y^\alpha, z) = \left(\sigma^\alpha e_\alpha, \rho^i_\alpha \sigma^\alpha \frac{\partial}{\partial x^i} + \left(\rho^i_\alpha y^\alpha \frac{\partial \sigma^\beta}{\partial x^i} + C^\beta_{\alpha\gamma} y^\alpha \sigma^\gamma\right)\frac{\partial}{\partial y^\beta}\right)$$

Definition 3 We say that $\sigma \in \Gamma(E)$ is an *infinitesimal symmetry* of the Herglotz Lagrangian L if the flow generated by σ^C on $E \times \mathbb{R}$ leaves L invariant. Infinitesimally, this means $\mathcal{L}^E_{\sigma^C} L = 0$, where $\mathcal{L}^E_{\sigma^C}$ is the Lie derivative acting on functions on E via the algebroid structure, i.e.,

$$\mathcal{L}^E_{\sigma^C} L = \rho_1(\sigma^C)L,$$

for any function L on $E \times \mathbb{R}$.

In local coordinates, the previous condition simply means that if σ is an infinitesimal symmetry then

$$\rho^i_\alpha \sigma^\alpha \frac{\partial L}{\partial x^i} + \rho^i_\alpha y^\alpha \frac{\partial \sigma^\beta}{\partial x^i}\frac{\partial L}{\partial y^\beta} + C^\beta_{\alpha\gamma} y^\alpha \sigma^\gamma \frac{\partial L}{\partial y^\beta} = 0. \tag{13}$$

Define the *Herglotz momentum* associated with σ by the local expression

$$J_\sigma = \frac{\partial L}{\partial y^\alpha}(x^i, y^\alpha, z)\, \sigma^\alpha(x^i).$$

Proposition 1 (Noether–Herglotz on a Lie algebroid) *Let $\sigma \in \Gamma(E)$ be an infinitesimal symmetry of L. Let $(x^i(t), y^\alpha(t), z(t))$ be a solution of the Herglotz equations. Then the associated momentum J_σ satisfies*

$$\dot{J}_\sigma = \frac{\partial L}{\partial z} J_\sigma. \tag{14}$$

Equivalently,

$$\frac{d}{dt}\left(e^{-\int_0^t \frac{\partial L}{\partial z}(\omega)\, d\omega} J_\sigma(t)\right) = 0$$

along the trajectories.

Hence the rescaled momentum $e^{-\int_0^t (\partial L/\partial z)(\omega)\, d\omega} J_\sigma$ is a time-dependent quantity conserved in the classical sense.

Proof By definition,

$$J_\sigma = \frac{\partial L}{\partial y^\alpha}\,\sigma^\alpha(x).$$

Differentiate with respect to t:

$$\dot{J}_\sigma = \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right)\sigma^\alpha + \frac{\partial L}{\partial y^\alpha}\frac{d}{dt}\big(\sigma^\alpha(x)\big).$$

Using the chain rule,

$$\frac{d}{dt}(\sigma^\alpha(x)) = \frac{\partial \sigma^\alpha}{\partial x^i}\dot{x}^i = \frac{\partial \sigma^\alpha}{\partial x^i}\,\rho^i{}_\beta y^\beta.$$

and substituting $\dot{x}^i = \rho^i{}_\beta y^\beta$. Inserting the Euler–Lagrange–Herglotz equations, we obtain

$$\dot{J}_\sigma = \left(\rho^i{}_\alpha \frac{\partial L}{\partial x^i} - C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial y^\alpha}\right)\sigma^\alpha + \frac{\partial L}{\partial y^\alpha}\frac{\partial \sigma^\alpha}{\partial x^i}\rho^i{}_\beta y^\beta.$$

Finally, the local expression (13) satisfied by σ due to the fact that it is an infinitesimal symmetry, implies that almost all the terms are zero except for the term containing the partial derivative with respect to the variable z which results in

$$\dot{J}_\sigma = \frac{\partial L}{\partial z}\,J_\sigma.$$

The integrating factor argument then shows that $e^{-\int_0^t (\partial L/\partial z)(\omega)\,d\omega} J_\sigma(t)$ is constant. □

Remark 4 In the conservative case $\partial L/\partial z = 0$, we recover the usual Noether theorem: J_σ is conserved. ◇

5 Coordinate-Free Formulation of the Euler–Lagrange–Herglotz Equations

In Sect. 3, we derived the Euler–Lagrange–Herglotz equations from admissible variations of admissible E-paths, without introducing any auxiliary connection. In this section, following the connection-based framework of Li-Stern-Tang and Hu-Stern, we provide a global rewriting of these equations using a TM-connection. This connection is used only as an auxiliary device to obtain a horizontal-vertical splitting of the dynamics. In order to obtain a coordinate-free formulation of the Herglotz equations, we briefly recall Lie algebroid connections and the Euler–Lagrange–Poincaré formalism developed in Li et al. (2017)- Hu and Stern (2024).

Let $(E \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid. A *TM-connection* on E is a linear connection

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E), \qquad (X, u) \mapsto \nabla_X u,$$

on the vector bundle $E \to M$. It induces two E-connection on E, denoted again by ∇ and $\overline{\nabla}$

$$\nabla_a u := \nabla_{\rho(a)} u, \qquad \overline{\nabla}_a u := \nabla_{\rho(u)} a + [a, u],$$

for $a \in E$, $u \in \Gamma(E)$. In addition, for each E-connection, a dual connection on the dual algebroid $\pi : E^* \to M$ denoted by ∇^* is induced satisfying

$$\langle \nabla^*_a p, u \rangle := \rho(a)\,[\langle p, u\rangle] - \langle p, \nabla_a u\rangle \quad \forall u \in \Gamma(E).$$

for $a \in E$ and $p \in \Gamma(E^*)$.

Given an admissible curve $a(t) \in E$ with base curve $x(t) \in M$, we write $a(t) \in E_{x(t)}$ and denote by

$$\nabla_{a(t)} \colon E_{x(t)} \longrightarrow E_{x(t)}$$

the covariant derivative along $a(t)$ induced by the E-connection ∇. For a curve $u(t) \in E_{x(t)}$ over the same base curve $x(t)$ whose values coincide with those of a time-dependent section $\eta(t, x(t))$ of $\Gamma(E)$, i.e., $u(t) = \eta(t, x(t))$, we set

$$\nabla_{a(t)} u(t) := \nabla_{a(t)} \eta(t, x(t)) + \dot{\eta}(t, x(t)) \in E_{x(t)},$$

where the covariant derivative is taken with respect to the E–connection.

Using a TM-connection on E each vector $X \in T_a E$ can be decomposed in a pair $(X_{\text{hor}}, X_{\text{ver}})$ with $X_{\text{hor}} \in TM$ and $X_{\text{ver}} \in E$, called the horizontal and vertical components of X. The vertical component belongs to the tangent to the fibers $X_{\text{ver}} \in T_a E_x \approx E_x$, while the horizontal component is in one-to-one correspondence with TM under the connection ∇ (see Section 3.2 in Fernandes (2002)). Similarly, a TM-connection determines a splitting of T^*E into horizontal and vertical subbundles. Any $\theta \in T^*_a E$ decomposes, relative to the TM–connection ∇, in the pair $(\theta_{\text{hor}}, \theta_{\text{ver}})$, with $\theta_{\text{ver}}(a) \in E^*_{\tau(a)}$ and $\theta_{\text{hor}}(a) \in T^*_{\tau(a)} M$ defined by

$$\langle \theta, X \rangle = \langle \theta_{\text{hor}}, X_{\text{hor}} \rangle, \qquad \text{for all } X \in T_a E \text{ such that } X_{\text{ver}} = 0$$

and

$$\langle \theta, X \rangle = \langle \theta_{\text{ver}}, X_{\text{ver}} \rangle, \qquad \text{for all } X \in T_a E \text{ such that } X_{\text{hor}} = 0.$$

Following Li et al. (2017); Hu and Stern (2024), given an admissible path $a(t) \in E$ over $x(t) \in M$ from $[0, 1]$, an admissible variation is of the form $X_{b,a}(t) \in T_{a(t)} E$, where b is a curve in E over $x(t)$ but not necessarily admissible such that $b(0) = b(1) = 0$. Relative to a TM-connection, the vertical component of $X_{b,a}$ is $\overline{\nabla}_a b$ and the horizontal component is $\rho(b)$. In Crainic and Fernandes (2003) it is shown that the space of admissible variations is independent of the choice of connection in the above definition.

When $L \colon E \to \mathbb{R}$ is an ordinary (non-Herglotz) Lagrangian, the Euler–Lagrange–Herglotz equations can be written in the connection-based form

$$\overline{\nabla}^*_{a(t)} p(t) - \rho^*\big(dL_{\text{hor}}(a(t))\big) = 0, \qquad p = dL_{\text{ver}}(a), \tag{15}$$

as shown in (Li et al., 2017, Section 3) and (Hu and Stern, 2024, Theorem 5.3). In Li et al. (2017) the notation $\overline{\nabla}^*$ does not denote the dual connection but the adjoint operator, causing a difference in the sign that we see in the equations.

5.1 Euler–Lagrange–Herglotz Equations

We now derive a global connection-based formulation of Herglotz variational principle on a Lie algebroid, in the spirit of Li et al. (2017).

Theorem 2 *Let $(E \to M, [\cdot, \cdot], \rho)$ be a Lie algebroid equipped with a TM–connection ∇, and let ∇^* be the induced E–connection on E^*. Let $L\colon E \times \mathbb{R} \to \mathbb{R}$ be a Herglotz-type Lagrangian and let $(x(t), a(t), z(t))$ be an admissible curve satisfying*

$$\dot{x}(t) = \rho(a(t)), \qquad \dot{z}(t) = L\big(a(t), z(t)\big).$$

Decompose the differential of L as

$$dL(a, z) = dL_{\text{hor}}(a, z) + dL_{\text{ver}}(a, z) + \frac{\partial L}{\partial z}(a, z)\, dz,$$

and define the momentum

$$p(t) := dL_{\text{ver}}\big(a(t), z(t)\big) \in E^*_{x(t)}.$$

Then an admissible curve $(x(t), a(t), z(t))$ satisfies the Euler–Lagrange–Herglotz equations if and only if it satisfies the intrinsic relation

$$\overline{\nabla}^*_{a(t)} p(t) - \rho^*\big(dL_{\text{hor}}(a(t), z(t))\big) = \frac{\partial L}{\partial z}\big(a(t), z(t)\big)\, p(t), \tag{16}$$

together with the Herglotz equation $\dot{z} = L(a, z)$.

Proof Following the argument in Li et al. (2017); Hu and Stern (2024) adapted to the contact case, take a variation $(X_{b,a}, \delta z)$ of a curve $(a(t), z(t))$ satisfying the Herglotz equation $\dot{z} = L(a(t), z(t))$. Then,

$$\frac{d}{dt}(\delta z) = \langle dL, (X_{b,a}, \delta z)\rangle = \langle dL_{\text{hor}}(a), \rho(b)\rangle + \langle dL_{\text{ver}}(a), \overline{\nabla}_a b\rangle + \frac{\partial L}{\partial z}\delta z.$$

Solving for $\delta z(t)$ using the integrating factor

$$\lambda(t) := \exp\left(- \int_0^t \frac{\partial L}{\partial z}(\omega)\, d\omega \right),$$

and also that $\delta z(0) = 0$ and $\lambda(t) \neq 0$, the stationarity condition $\delta z(T) = 0$ is equivalent to

$$\int_0^T \lambda(t) \left(\langle dL_{\text{hor}}(a), \rho(b) \rangle + \langle dL_{\text{ver}}(a), \overline{\nabla}_a b \rangle \right) dt = 0.$$

Equivalently, after integrating by parts the therm with $\overline{\nabla}_a b$ and using the fact that b vanishes in the boundaries, we obtain

$$\int_0^T \left(\langle \lambda(t) \rho^* dL_{\text{hor}}(a), b \rangle - \langle \overline{\nabla}_a^* \left(\lambda(t) dL_{\text{ver}}(a) \right), b \rangle \right) dt = 0.$$

The second term in the integrand satisfies

$$\overline{\nabla}_a^* \left(\lambda(t) dL_{\text{ver}}(a) \right) = \lambda(t) \overline{\nabla}_a^* \left(dL_{\text{ver}}(a) \right) + \dot{\lambda}(t) dL_{\text{ver}}(a).$$

Hence, we obtain

$$\int_0^T \left(\lambda(t) \langle \rho^* dL_{\text{hor}}(a) - \overline{\nabla}_a^* \left(dL_{\text{ver}}(a) \right) + \frac{\partial L}{\partial z} dL_{\text{ver}}(a), b \rangle \right) dt = 0.$$

for arbitrary variations b. Thus, from the fundamental theorem of calculus of variations we must have

$$\rho^* dL_{\text{hor}}(a) - \overline{\nabla}_a^* \left(dL_{\text{ver}}(a) \right) + \frac{\partial L}{\partial z} dL_{\text{ver}}(a) = 0.$$

The converse follows by reversing the above computations, showing that then the integral vanishes for all admissible variations.

Alternatively, expanding (16) in a local frame (see Lemma 1 in the Appendix) yields the local Euler–Lagrange–Herglotz equations. □

Remark 5 An important remark is that the local expression of the equations of motion is independent of the choice of TM-connection in the previous theorem, as it is pointed out in Hu and Stern (2024) (see also 1 in the Appendix).

Remark 6 It is important to note that connection-free intrinsic formulations of variational classical mechanics on algebroids are available in the literature, though they have not yet been adapted to the setting of Herglotz variational problem. In particular, the framework of Grabowska and Grabowski (2008) encodes the algebroid structure by a double vector bundle morphism and constructs admissible variations intrinsically, without introducing an auxiliary TM-connection. In the present paper, however, we adopt the connection-based viewpoint of Li et al. (2017) and Hu and Stern (2024) because it provides a convenient horizontal-vertical splitting of the dynamics, particularly useful for explicit reduced equations and for future developments toward structure-preserving contact-type integrators. As shown in Appendix A, the resulting local equations are independent of the chosen connection, so the connection is used here only as an auxiliary device and not as part of the geometric data of the Herglotz variational problem.

6 Hamilton–Pontryagin–Herglotz Principle on a Lie Algebroid

In the Hamilton–Pontryagin formulation of Lagrangian mechanics on a Lie algebroid (Li et al. 2017), one considers triples of paths (a, v, p) in E and E^* together with a base path $x(t)$, and imposes the kinematic constraint $a = v$ through a Lagrange multiplier p. In the Herglotz setting, the action is replaced by a scalar variable $z(t)$ satisfying a contact-type evolution equation.

Following (Li et al., 2017, Def. 5.3), an (E, E^*)–path consists of:

(i) an E–path a over a base curve $x: I \to M$;
(ii) an arbitrary curve v on E over the same base curve x but not necessarily an E-path;
(iii) a curve p on E^* over x.

Admissible variations of (a, v, p, z) are the same as in Li et al. (2017): $\delta a = X_{b,a}$, δz, δv and δp are any variations satisfying $\tau_*(\delta v) = \pi_*(\delta p) = \rho(b)$

Let $L: E \times \mathbb{R} \to \mathbb{R}$ be a Herglotz Lagrangian. Instead of an action functional, we define the *Pontryagin–Herglotz action variable* $z(t)$ by

$$\dot{z}(t) = L\big(v(t), z(t)\big) + \langle p(t),\, a(t) - v(t)\rangle. \tag{17}$$

Given a 1–parameter family (a_s, v_s, p_s, z_s) of variations, the *Hamilton–Pontryagin–Herglotz principle* is

$$\left.\frac{d}{ds}\right|_{s=0} z_s(T) = 0,$$

where z_s solves the perturbed equation obtained from (17).

Definition 4 A curve (a, v, p, z) satisfies the *Hamilton–Pontryagin–Herglotz principle* if for every admissible variation $(\delta a, \delta v, \delta p, \delta z)$ with $\delta x(0) = \delta x(T) = 0$ and $\delta z(0) = 0$ one has $\left.\frac{d}{ds}\right|_{s=0} z_s(T) = 0.$

Theorem 3 (Implicit Euler–Lagrange–Poincaré–Herglotz equations) *If (a, v, p, z) satisfies the Hamilton–Pontryagin–Herglotz principle, then:*

$$\begin{aligned}
&\text{(i)}\quad a(t) \textit{ is an } E\textit{–path: } \dot{x} = \rho(a),\\
&\text{(ii)}\quad a(t) = v(t),\\
&\text{(iii)}\quad p(t) = dL_{\text{ver}}\big(a(t), z(t)\big),\\
&\text{(iv)}\quad \overline{\nabla}^*_a p(t) - \rho^* dL_{\text{hor}}(a(t), z(t)) - \frac{\partial L}{\partial z}(a(t), z(t))p(t) = 0,\\
&\text{(v)}\quad \dot{z}(t) = L(a(t), z(t)).
\end{aligned} \tag{18}$$

Conversely, any curve satisfying (i)–(v) satisfies the Hamilton–Pontryagin–Herglotz principle and projects to a Herglotz extremal.

Proof Take an admissible variation $(X_{b,a}, \delta v, \delta p, \delta z)$ of a curve $(a(t), v(t), p(t), z(t))$ satisfying the Herglotz equation $\dot{z} = L(a(t), z(t)) + \langle p(t),\, a(t) - v(t)\rangle$. Then,

$$\begin{aligned}\frac{d}{dt}(\delta z) &= \langle dL, (\delta v, \delta z)\rangle + \langle \delta p, a - v\rangle + \langle p, X_{b,a}\rangle - \langle p, \delta v\rangle \\ &= \langle dL_{\mathrm{hor}}(a), \rho(b)\rangle + \langle dL_{\mathrm{ver}}(a), \delta v_{\mathrm{ver}}\rangle + \frac{\partial L}{\partial z}\delta z \\ &\quad + \langle \delta p_{\mathrm{ver}}, a - v\rangle + \langle \rho(b), a - v\rangle + \langle p, \overline{\nabla}_a b - \delta v_{\mathrm{ver}}\rangle.\end{aligned}$$

Solving for $\delta z(t)$ using the integrating factor

$$\lambda(t) := \exp\left(-\int_0^t \frac{\partial L}{\partial z}(\omega)\, d\omega\right),$$

and also that $\delta z(0) = 0$ and $\lambda(t) \neq 0$, the stationarity condition $\delta z(T) = 0$ is equivalent to

$$\begin{aligned}\int_0^T \lambda(t)\Big(\langle dL_{\mathrm{hor}}, \rho(b)\rangle + \langle dL_{\mathrm{ver}}, \delta v_{\mathrm{ver}}\rangle + \langle \delta p_{\mathrm{ver}}, a - v\rangle \\ + \langle \rho(b), a - v\rangle + \langle p, \overline{\nabla}_a b - \delta v_{\mathrm{ver}}\rangle\Big)\, dt = 0.\end{aligned}$$

Equivalently,

$$\begin{aligned}\int_0^T \Big(\langle \lambda(t)\rho^* dL_{\mathrm{hor}}(a) + \lambda(t)\rho^*(a - v) - \overline{\nabla}_a^* (\lambda(t) p)\, , b\rangle \\ + \langle dL_{\mathrm{ver}} - p, \delta v_{\mathrm{ver}}\rangle + \langle \delta p_{\mathrm{ver}}, a - v\rangle\Big)\, dt = 0.\end{aligned}$$

The third term in the integrand satisfies

$$\overline{\nabla}_a^* (\lambda(t) p(t)) = \lambda(t)\overline{\nabla}_a^* p(t) + \dot{\lambda}(t) p(t).$$

Hence, we obtain

$$\begin{aligned}\int_0^T \Big(\lambda(t)\langle \rho^* dL_{\mathrm{hor}}(a) + \rho^*(a - v) - \overline{\nabla}_a^* p + \frac{\partial L}{\partial z} p, b\rangle \\ + \langle dL_{\mathrm{ver}} - p, \delta v_{\mathrm{ver}}\rangle + \langle \delta p_{\mathrm{ver}}, a - v\rangle\Big)\, dt = 0.\end{aligned}$$

for arbitrary variations b, δv_{ver} and δp_{ver}. Thus, from the fundamental theorem of calculus of variations we must have

$$\rho^* dL_{\mathrm{hor}}(a) - \overline{\nabla}_a^* p + \frac{\partial L}{\partial z} p = 0, \quad dL_{\mathrm{ver}} = p, \quad a = v.$$

Conversely, if we reverse all the arguments, the integrand in $\delta z(T)$ vanishes identically, so the Hamilton–Pontryagin–Herglotz principle holds. □

Corollary 1 (Intrinsic Euler–Lagrange–Poincaré–Herglotz equations) *Let* $(x(t), a(t), z(t))$ *satisfy* $\dot{x} = \rho(a)$ *and* $\dot{z} = L(a, z)$ *and define* $p = dL_{\mathrm{ver}}(a, z)$*. Then* (x, a, z) *satisfies the local Euler–Lagrange–Herglotz equations if and only if* p *satisfies the intrinsic equation*

$$\overline{\nabla}_a^* p(t) - \rho^* dL_{\mathrm{hor}}(a(t), z(t)) - \frac{\partial L}{\partial z}(a(t), z(t))p(t) = 0, \tag{19}$$

Proof By Theorem 2, for any admissible curve $(x(t), a(t), z(t))$ with $\dot{x} = \rho(a)$ and $\dot{z} = L(a, z)$, the local Euler–Lagrange–Herglotz equations are equivalent to the intrinsic relation

$$\overline{\nabla}_a^* p(t) - \rho^* dL_{\mathrm{hor}}(a(t), z(t)) - \frac{\partial L}{\partial z}(a(t), z(t))p(t) = 0,$$

where $p(t) = dL_{\mathrm{ver}}(a(t), z(t))$. This is precisely the statement of the corollary. □

Remark 7 The classical Hamilton–Pontryagin principle on a Lie algebroid (Li et al. 2017) is based on the variational stationarity of an action functional. In the Herglotz setting, no action functional exists: the quantity $z(t)$ is defined dynamically by the contact-type evolution equation $\dot{z} = L(v, z) + \langle p, a - v \rangle$, and variational stationarity is imposed on the terminal value $z(T)$. This single modification produces a contact deformation of all geometric structures involved.

Thus the Hamilton–Pontryagin–Herglotz principle provides a natural contact-type extension of the classical Pontryagin variational principle on Lie algebroids, retaining its geometric structure while incorporating dissipation and nonequilibrium effects in a covariant manner. ◇

7 Examples

In this section, we illustrate some applications of our previous results in specific examples of dissipative mechanical systems on different Lie algebroids.

7.1 Euler–Lagrange–Herglotz Equations

Example 1 We first recall some of the most basic examples of Euler–Lagrange–Herglotz equations whose study proceeds the quest for the Lie algebroid formulation of the contact setting.

Let $E = TQ$ be the tangent bundle of a configuration manifold Q. In this case, we choose local coordinates (q^i) and induced coordinates $(q^i, \dot{q}^i)$ on TQ, i.e., $x^i = q^i$, $y^i = \dot{q}^i$, and the Lie algebroid structure is trivial. Then the Lagrangian is $L =$

$L(q, \dot{q}, z)$, and the Euler–Lagrange–Herglotz equations (9) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^i} = 0, \quad \dot{z} = L(q, \dot{q}, z),$$

which are the classical Euler–Lagrange–Herglotz equations.

Let $E = \mathfrak{g}$ be a Lie algebra, regarded as a Lie algebroid over a point. Thus $M = \{*\}$, the anchor vanishes, and an admissible curve in E is simply a curve $t \mapsto \xi(t) \in \mathfrak{g}$. Choose a basis $\{e_\alpha\}$ of $\mathfrak{g}$ with brackets $[e_\alpha, e_\beta] = C_{\alpha\beta}^{\gamma} e_\gamma$. A Herglotz Lagrangian on $\mathfrak{g}$ is a function $\ell(\xi, z)$ with $(\xi, z) \in \mathfrak{g} \times \mathbb{R}$. Writing $\xi = \xi^\alpha e_\alpha$, the Euler–Lagrange–Herglotz equations reduce to

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi^\alpha}\right) + C_{\alpha\beta}^{\gamma}\, \xi^\beta \frac{\partial \ell}{\partial \xi^\gamma} = \frac{\partial \ell}{\partial z}\frac{\partial \ell}{\partial \xi^\alpha}, \qquad \dot{z} = \ell(\xi, z).$$

Identifying the covector

$$\frac{\partial \ell}{\partial \xi} := \frac{\partial \ell}{\partial \xi^\alpha} e^\alpha \in \mathfrak{g}^*,$$

the equation takes the intrinsic Euler–Poincaré–Herglotz form Anahory et al. (2024)

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right) + \mathrm{ad}_\xi^*\left(\frac{\partial \ell}{\partial \xi}\right) = \frac{\partial \ell}{\partial z}\frac{\partial \ell}{\partial \xi} \qquad \dot{z} = \ell(q, \xi, z).$$

Example 2 (Euler–Poincaré–Herglotz equations on an action Lie algebroid)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on a manifold Q from the left. The associated action Lie algebroid is $E = Q \times \mathfrak{g} \to Q$, with anchor and bracket given by $\rho(q, \xi) = \xi_Q(q)$, $[(q, \xi), (q, \eta)] = (q, [\xi, \eta])$, where ξ_Q denotes the infinitesimal generator of $\xi \in \mathfrak{g}$.

Choose a basis $\{e_\alpha\}$ of $\mathfrak{g}$, with $[e_\alpha, e_\beta] = C_{\alpha\beta}^{\gamma} e_\gamma$, and write $\xi = \xi^\alpha e_\alpha$. The anchor in local coordinates reads

$$\rho(q, e_\alpha) = \rho_\alpha^i(q)\, \frac{\partial}{\partial q^i} = (e_\alpha)_Q(q),$$

so that the admissibility condition for a curve $(q(t), \xi(t))$ becomes $\dot{q}^i(t) = \rho_\alpha^i\big(q(t)\big)\, \xi^\alpha(t)$.

Taking the Herglotz Lagrangian on the action Lie algebroid $\ell(q, \xi, z)$ with $(q, \xi, z) \in Q \times \mathfrak{g} \times \mathbb{R}$, the Euler–Lagrange–Herglotz equations (9) become

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi^\alpha}\right) + C_{\alpha\beta}^{\gamma}\, \xi^\beta \frac{\partial \ell}{\partial \xi^\gamma} - \rho_\alpha^i(q)\, \frac{\partial \ell}{\partial q^i} = \frac{\partial \ell}{\partial z}\frac{\partial \ell}{\partial \xi^\alpha}, \qquad \dot{z} = \ell(q, \xi, z), \tag{20}$$

together with the kinematic relation $\dot{q}^i = \rho_\alpha^i(q)\, \xi^\alpha$.

Let $J : T^*Q \to \mathfrak{g}^*$ be the standard momentum map for the cotangent-lifted action, characterized by $\big\langle J(q, p_q), \eta \big\rangle = \big\langle p_q, \eta_Q(q) \big\rangle$ with $\eta \in \mathfrak{g}$. Then, (20) can be written intrinsically as the Euler–Poincaré–Herglotz equations on the action Lie algebroid

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right) + \mathrm{ad}^*_{\xi}\left(\frac{\partial \ell}{\partial \xi}\right) = J\left(\frac{\partial \ell}{\partial q}\right) + \frac{\partial \ell}{\partial z}\,\frac{\partial \ell}{\partial \xi}, \tag{21}$$

with

$$\dot{q} = \xi_Q(q), \qquad \dot{z} = \ell(q, \xi, z).$$

Example 3 *(Lagrange–Poincaré–Herglotz equations on the Atiyah algebroid)* Let $\pi : Q \to M := Q/G$ be a principal G-bundle, with $\dim M = n$ and $\dim G = d$. The quotient $\widehat{TQ} = TQ/G \to M$ is the Atiyah algebroid. Choose a local trivialization of Q and let $\{e_i, \widehat{e}_A\}$ be the induced local G-invariant frame of TQ/G (see Simoes et al. 2024a, b, 1923 for details). Then the anchor and bracket are

$$\rho(e_i) = \frac{\partial}{\partial q^i}, \qquad \rho(\widehat{e}_A) = 0,$$

$$[e_i, e_j] = -\mathcal{B}^A_{ij}\,\widehat{e}_A, \qquad [e_i, \widehat{e}_A] = c^{\,C}_{AB}\,\mathcal{A}^B_i\,\widehat{e}_C, \qquad [\widehat{e}_A, \widehat{e}_B] = c^{\,C}_{AB}\,\widehat{e}_C,$$

where $\mathcal{A}^A_i$ and $\mathcal{B}^A_{ij}$ are the local coefficients of the principal connection and its curvature, and $c^{\,C}_{AB}$ are the structure constants of $\mathfrak{g}$.

Let $L = L(q^i, \dot{q}^i, v^A, z) : \widehat{TQ} \times \mathbb{R} \to \mathbb{R}$ be the reduced Herglotz Lagrangian induced by a G-invariant Lagrangian on $TQ \times \mathbb{R}$. Applying Theorem 1 to this Lie algebroid yields the Lagrange–Poincaré–Herglotz equations:

$$\frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) = \frac{\partial L}{\partial v^A}\left(\mathcal{B}^A_{ij}\,\dot{q}^i + c^{\,A}_{DB}\,\mathcal{A}^B_j\,v^D\right) - \frac{\partial L}{\partial z}\,\frac{\partial L}{\partial \dot{q}^j}, \qquad j = 1, \ldots, n, \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^B}\right) = \frac{\partial L}{\partial v^A}\left(c^{\,A}_{DB}\,v^D - c^{\,A}_{DB}\,\mathcal{A}^D_i\,\dot{q}^i\right) + \frac{\partial L}{\partial z}\,\frac{\partial L}{\partial v^B}, \qquad B = 1, \ldots, d, \tag{23}$$

$$\dot{z} = L(q^i, \dot{q}^i, v^A, z). \tag{24}$$

These are the *Lagrange–Poincaré–Herglotz equations* on the Atiyah algebroid. They coincide with the reduced contact Lagrange–Poincaré equations previously obtained in Simoes et al. (2024a, b, 1923).

Example 4 *(Dissipative Wong's equations)* To illustrate the above Lagrange–Poincaré–Herglotz equations, we consider a dissipative version of the classical Wong equations. These equations arise, for instance, in the dynamics of a charged particle moving in a Yang–Mills field and in the geometric analysis of the falling cat problem (see Cendra et al. 2001 and references therein). In both situations, the reduced dynamics is governed by the Atiyah algebroid associated with a principal G-bundle, and the internal variables evolve in the adjoint bundle $Ad(Q)$.

The Herglotz framework provides a natural geometric mechanism to incorporate dissipative effects in these models. A term linear in the contact variable z produces, after reduction, a controlled decay of the horizontal and vertical momenta, in accordance with the notion of dissipated quantities introduced in De León et al. (2021). Physically, this allows one to model frictional or damping interactions with a surrounding medium (as in the case of a particle in a non-ideal Yang–Mills environment) and, in the context of the falling cat, the gradual loss of internal rotational energy due to elastic or viscoelastic couplings inside the body. In this way, Herglotz-type dissipation becomes compatible with the symmetry reduction and yields a gauge-invariant dissipative extension of Wong's equations.

Let (M, g_M) be a Riemannian manifold, G a compact Lie group endowed with a bi-invariant Riemannian metric κ, and let $\pi : Q \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$. Let $\mathcal{A} : TQ \to \mathfrak{g}$ be a principal connection with curvature $B : TQ \oplus TQ \to \mathfrak{g}$. Using $\mathcal{A}$ one gets an identification $T_qQ \cong T_{\pi(q)}M \oplus \mathfrak{g}$ with $q \in Q$, and the metrics g_M and κ jointly induce a G-invariant Riemannian metric g_Q on Q.

We consider the *kinetic Herglotz Lagrangian* $L : TQ \times \mathbb{R} \to \mathbb{R}$ given by

$$L(v_q, z) = \frac{1}{2}\Big(\kappa_e\big(\mathcal{A}(v_q), \mathcal{A}(v_q)\big) + g_{M,\pi(q)}\big((T_q\pi)(v_q), (T_q\pi)(v_q)\big)\Big) - \gamma\, z, \quad (25)$$

for $v_q \in T_qQ$, where $e \in G$ is the identity and $\gamma > 0$ is a dissipation parameter. Clearly L is hyperregular and G-invariant.

Since g_Q is G-invariant, it induces a fiber metric $g_{TQ/G}$ on the Atiyah algebroid $\widehat{TQ} = TQ/G \to M = Q/G$. The reduced Herglotz Lagrangian $L_{\mathrm{red}} : \widehat{TQ} \times \mathbb{R} \to \mathbb{R}$ is then the kinetic energy of $g_{TQ/G}$ minus the linear term in z:

$$L_{\mathrm{red}}([v_q], z) = \frac{1}{2}\Big(\kappa_e\big(\mathcal{A}(v_q), \mathcal{A}(v_q)\big) + g_{M,\pi(q)}\big((T_q\pi)(v_q), (T_q\pi)(v_q)\big)\Big) - \gamma\, z, \quad (26)$$

for $[v_q] \in TQ/G$. The Legendre transform associated with L_{red} is the bundle isomorphism $([v_q], s) \mapsto \big(\flat_{g_{TQ/G}}([v_q]), s\big)$, where $\flat_{g_{TQ/G}} : TQ/G \to T^*Q/G$ is induced by the fiber metric $g_{TQ/G}$.

Let us choose a local trivialization $\pi^{-1}(U) \simeq U \times G$ of the principal bundle $\pi : Q \to M$, where $U \subset M$ has local coordinates (q^i). Let $\{\xi_A\}$ be a basis of $\mathfrak{g}$ with structure constants $c_{AB}^{\,D}$, let $\mathcal{A}_i^A$ and $\mathcal{B}_{ij}^A$ be the local components of $\mathcal{A}$ and its curvature B, and write $\kappa_e = \kappa_{AB}\, \xi^A \otimes \xi^B$ and $g_M = g_{ij}\, dq^i \otimes dq^j$, with $\{\xi^A\}$ the dual basis of $\{\xi_A\}$. The bi-invariance of κ implies the identity $c_{AB}^{\,D}\,\kappa_{DE} = c_{AE}^{\,D}\,\kappa_{DB}$.

Denote by $\{e_i, \widehat{\xi}_A\}$ the associated G-invariant local frame on Q and by $(q^i, \dot{q}^i, v^A, z)$ the induced local fibred coordinates on $\widehat{TQ} \times \mathbb{R}$. In these coordinates the reduced Lagrangian (26) becomes

$$L_{\mathrm{red}}(q^i, \dot{q}^i, v^A, z) = \frac{1}{2}\left(\kappa_{AB} v^A v^B + g_{ij}\dot{q}^i\dot{q}^j\right) - \gamma\, z. \quad (27)$$

The Hessian of L_{red} with respect to $(\dot{q}^i, v^A)$ is $W_{L_{\text{red}}} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \kappa_{AB} \end{pmatrix}$, hence L_{red} is hyperregular.

Applying the Lagrange–Poincaré–Herglotz equations (22)–(24) to (27), we obtain the *dissipative Wong equations* on the base M:

$$\frac{\partial g_{im}}{\partial q^j}\,\dot{q}^i\dot{q}^m - \frac{\partial g_{ij}}{\partial q^k}\,\dot{q}^k\dot{q}^i - g_{ij}\,\ddot{q}^i = \kappa_{AB}\,v^B\Big(\mathcal{B}^A_{ij}\,\dot{q}^i + c^A_{DB}\,\mathcal{A}^B_j\,v^D\Big) + \gamma\,g_{ij}\,\dot{q}^i, \tag{28}$$

$$\kappa_{AB}\,\dot{v}^A = \kappa_{AE}v^E\Big(c^A_{DB}\,v^D - c^A_{DB}\,\mathcal{A}^D_i\,\dot{q}^i\Big) - \gamma\,\kappa_{AB}\,v^A, \tag{29}$$

together with the contact equation $\dot{z} = L_{\text{red}}(q^i, \dot{q}^i, v^A, z)$.

7.2 Noether Symmetries

Example 5 *(Rigid body with dissipated body angular momentum)* We next consider a rigid body with configuration Lie group $G = \mathrm{SO}(3)$ and Lie algebra $\mathfrak{g} \simeq \mathbb{R}^3$. We work on the Lie algebra viewed as a Lie algebroid over a point, $E = \mathfrak{g}$. Let $\mathbb{I}\colon \mathfrak{g} \to \mathfrak{g}^*$ be the inertia operator. We choose body angular velocity $\xi \in \mathfrak{g}$ and define for $\gamma > 0$ the Herglotz-type Lagrangian (see Anahory et al. 2024)

$$\ell(\xi, z) = \frac{1}{2}\,\langle \mathbb{I}\xi, \xi\rangle - \gamma\,z.$$

The Euler–Poincaré–Herglotz equations on $\mathfrak{g}$ read

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right) + \mathrm{ad}^*_\xi\left(\frac{\partial \ell}{\partial \xi}\right) = \frac{\partial \ell}{\partial z}\,\frac{\partial \ell}{\partial \xi}, \qquad \dot{z} = \ell(\xi, z).$$

Identifying the body angular momentum $\mu := \partial\ell/\partial\xi = \mathbb{I}\xi$, we obtain

$$\dot{\mu} + \mathrm{ad}^*_\xi\,\mu = -\gamma\,\mu.$$

Suppose now that the inertia tensor is axially symmetric about the body e_3-axis, so that the Lagrangian is invariant under rotations generated by e_3. In the conservative case ($\gamma = 0$), Noether's theorem implies that the component $\langle \mu, e_3\rangle$ of the body angular momentum is conserved (see Marsden and Ratiu 2013 for instance).

In the Herglotz setting, Noether–Herglotz Proposition 1 gives instead

$$\frac{d}{dt}J_\sigma = \frac{\partial \ell}{\partial z}\,J_\sigma = -\gamma\,J_\sigma,$$

where $J_\sigma(t) = \langle \mu(t), e_3\rangle$ is the momentum associated with the symmetry $\sigma = e_3$. Thus

$$J_\sigma(t) = J_\sigma(0)\,e^{-\gamma t} \quad \text{and} \quad \widetilde{J}_\sigma(t) := e^{\gamma t}\,J_\sigma(t) \text{ is conserved.}$$

From the viewpoint of De León et al. (2021), $\widetilde{J}_\sigma$ is a *dissipated quantity*: the usual body component of angular momentum decays exponentially, but after multiplication by the integrating factor determined by the Herglotz term it becomes an invariant of the motion.

Example 6 *(Dissipated energy and momenta in Wong-type systems)* We finally revisit the dissipative Wong equations of Example 4 from the viewpoint of energy balance and dissipated quantities for particles with symmetries.

Let $L_{\text{red}} : \widehat{TQ} \times \mathbb{R} \to \mathbb{R}$ be the reduced Herglotz Lagrangian on the Atiyah algebroid $\widehat{TQ} = TQ/G \to M$, of the form

$$L_{\text{red}}(q^i, \dot{q}^i, v^A, z) = \frac{1}{2}\big(\kappa_{AB}\, v^A v^B + g_{ij}\, \dot{q}^i \dot{q}^j\big) - \gamma\, z, \qquad \gamma > 0,$$

where g_{ij} and κ_{AB} encode the metrics on the base M and the Lie algebra $\mathfrak{g}$, respectively.

The corresponding generalized energy is

$$\begin{aligned} E(q, \dot{q}, v, z) &= \left\langle \frac{\partial L_{\text{red}}}{\partial \dot{q}}, \dot{q} \right\rangle + \left\langle \frac{\partial L_{\text{red}}}{\partial v}, v \right\rangle - L_{\text{red}}(q, \dot{q}, v, z) \\ &= \frac{1}{2}\big(\kappa_{AB}\, v^A v^B + g_{ij}\, \dot{q}^i \dot{q}^j\big) + \gamma z, \end{aligned}$$

that is, the total kinetic energy. Since $\partial L_{\text{red}}/\partial z = -\gamma$, the energy balance law implies

$$\dot{E} = \frac{\partial L_{\text{red}}}{\partial z}\, E = -\gamma\, E,$$

and hence

$$E(t) = E(0)\, e^{-\gamma t}, \qquad \widetilde{E}(t) := e^{\gamma t}\, E(t) \text{ is conserved.}$$

Thus the kinetic energy of the Wong system decays exponentially, while the rescaled quantity $\widetilde{E}$ is a dissipated invariant.

A second family of dissipated quantities arises from the internal symmetry. Consider the G-action on Q and the induced G-action on $\widehat{TQ}$, and let $\sigma \in \Gamma(E)$ be the section corresponding to a fixed element $\eta \in \mathfrak{g}$. In the non-dissipative case, Noether's theorem yields conservation of the associated momentum $\langle \frac{\partial L_{\text{red}}}{\partial v}, \eta \rangle$ Cendra et al. (2001). In the Herglotz case, Proposition 1 gives

$$\frac{d}{dt} J_\eta = \frac{\partial L_{\text{red}}}{\partial z}\, J_\eta = -\gamma\, J_\eta, \qquad J_\eta(t) := \left\langle \frac{\partial L_{\text{red}}}{\partial v}, \eta \right\rangle = \kappa_{AB}\, v^A \eta^B,$$

and hence

$$J_\eta(t) = J_\eta(0)\, e^{-\gamma t}, \qquad \widetilde{J}_\eta(t) := e^{\gamma t}\, J_\eta(t) \text{ is conserved.}$$

Therefore, both the kinetic energy and the momenta of the Wong system become *dissipated quantities*: they decay exponentially along solutions, while their appropriately rescaled versions are genuine invariants of the dissipative dynamics on the Atiyah algebroid.

Example 7 *(Thermoviscous rigid body as an Euler–Poincaré–Herglotz system)*
We consider a simple finite-dimensional thermomechanical system: a rigid body with configuration space $Q = SO(3)$, whose dynamics is affected by a linear viscous torque and coupled to an entropy variable $S(t)$. The Herglotz variable z will be interpreted as the entropy S.

The Euler–Poincaré–Herglotz formulation of a thermoviscous rigid body provides a symmetry–reduced counterpart of the finite-dimensional thermodynamic systems with linear dissipation studied in Gay-Balmaz and Yoshimura (2017), and it parallels the rigid body thermomechanical models on Lie groups of Couéraud and Gay-Balmaz (2020), where viscous torques and entropy production give rise to dissipative Euler-Poincaré-Herglotz dynamics.

Let $G = SO(3)$ act on $Q = SO(3)$ by left multiplication. The associated action Lie algebroid is $E = Q \times \mathfrak{so}(3) \to Q,\ \tau(R, \xi) = R$. Its structure is:

- The anchor map $\rho\colon Q \times \mathfrak{so}(3) \to TQ$, $\rho(R, \xi) = \xi_Q(R)$, where ξ_Q is the infinitesimal generator of the left action: $\xi_Q(R) = \frac{d}{dt}\big|_{t=0}\big(\exp(t\xi)\, R\big)$.
- The bracket on sections is induced from the Lie bracket on $\mathfrak{so}(3)$:

$$[\sigma_1, \sigma_2](R) = \big(R, [\xi_1, \xi_2]\big), \quad \text{if } \sigma_i(R) = (R, \xi_i),\ i = 1, 2.$$

Choose a basis $\{e_\alpha\}_{\alpha=1}^3$ of $\mathfrak{so}(3)$ with structure constants $[e_\alpha, e_\beta] = C_{\alpha\beta}^{\gamma} e_\gamma$, and write any $\xi \in \mathfrak{so}(3)$ as $\xi = \xi^\alpha e_\alpha$. A curve in the algebroid is then $(R(t), \xi(t))$, with coordinates $\xi^\alpha(t)$. The admissibility condition $\dot{R} = \rho(R, \xi)$ reads $\dot{R}(t) = \xi_Q\big(R(t)\big) = R(t)\,\widehat{\xi(t)}$, where $\widehat{\xi} \in \mathfrak{so}(3)$ is the skew-symmetric matrix corresponding to $\xi \in \mathbb{R}^3$. Thus admissible curves encode the usual rigid body kinematics in body coordinates.

Let $I\colon \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ be a positive definite inertia operator, and let $U\colon Q \to \mathbb{R}$ be a potential energy (e.g., gravitational). We interpret the scalar variable $z(t)$ as the entropy $S(t)$ of the body. Fix a constant temperature parameter $T_0 > 0$, and a viscous coefficient $\gamma > 0$.

We define the Herglotz-type Lagrangian

$$L\colon E \times \mathbb{R} \to \mathbb{R}, \qquad L(R, \xi, S) = \frac{1}{2}\,\langle I\xi, \xi\rangle - U(R) - T_0\, S \;+\; \frac{\gamma}{2T_0}\,\|\xi\|^2.$$

Note that the term $\frac{1}{2}\langle I\xi, \xi\rangle - U(R)$ is the usual mechanical Lagrangian of the rigid body, $-T_0 S$ encodes the coupling to entropy at temperature T_0 (Legendre-type term $-TS$), and the term $\frac{\gamma}{2T_0}\|\xi\|^2$ is a simple quadratic dissipation potential, scaled by $1/T_0$ so that the corresponding entropy production will be proportional to the dissipation rate.

The entropy satisfies $\dot{S}(t) = L\big(R(t), \xi(t), S(t)\big)$. For a Herglotz Lagrangian

$\ell(\xi, S) = L(R, \xi, S)$(here ℓ does not depend explicitly on R if U is left-invariant), the Euler–Poincaré–Herglotz equations on the Lie algebra $\mathfrak{so}(3)$ read (cf. Example 2 with trivial base dependence)

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \xi}\right) + \mathrm{ad}^*_\xi\left(\frac{\partial \ell}{\partial \xi}\right) = \frac{\partial \ell}{\partial S}\,\frac{\partial \ell}{\partial \xi}, \qquad \dot{S} = \ell(\xi, S), \tag{30}$$

together with the kinematic relation

$$\dot{R} = R\widehat{\xi}.$$

In our case,
$$\frac{\partial \ell}{\partial \xi} = I\xi + \frac{\gamma}{T_0}\,\xi =: \Pi(\xi), \qquad \frac{\partial \ell}{\partial S} = -T_0.$$

Thus (30) becomes

$$\frac{d}{dt}\Pi(\xi) + \mathrm{ad}^*_\xi\,\Pi(\xi) = -\,T_0\,\Pi(\xi),\ \dot{S} = \frac{1}{2}\langle I\xi, \xi\rangle - U(R) - T_0\,S + \frac{\gamma}{2T_0}\|\xi\|^2. \tag{31}$$

The first equation is an *Euler–Poincaré equation with linear damping* on the momentum $\Pi(\xi)$, while the second encodes the entropy production.

The Herglotz energy

$$E(R, \xi, S) = \left\langle \frac{\partial L}{\partial \xi}, \xi \right\rangle - L = \langle I\xi, \xi\rangle + \frac{\gamma}{T_0}\|\xi\|^2 - \left(\frac{1}{2}\langle I\xi, \xi\rangle - U(R) - T_0 S + \frac{\gamma}{2T_0}\|\xi\|^2\right)$$

simplifies to

$$E(R, \xi, S) = \frac{1}{2}\langle I\xi, \xi\rangle + \frac{\gamma}{2T_0}\|\xi\|^2 + U(R) + T_0 S.$$

By the general energy balance law for Herglotz systems,

$$\dot{E} = \frac{\partial L}{\partial S}\,E = -T_0\,E,$$

so $E(t)$ decays exponentially:

$$E(t) = E(0)\,e^{-T_0 t}.$$

Energy decrease is not violating the first law of Thermodynamics since we are describing a non-isolated physical system loosing energy to its surrondings due to the dissipative term. At the same time, the entropy production law

$$\dot{S} = \frac{1}{2}\langle I\xi, \xi\rangle - U(R) - T_0 S + \frac{\gamma}{2T_0}\|\xi\|^2$$

shows that, for suitable regimes (e.g., U bounded below, large damping or near equilibrium), the entropy S increases, and its growth is driven by the dissipative term $\frac{\gamma}{2T_0}\|\xi\|^2$. In this sense, the Herglotz term realizes a finite-dimensional, symmetry-reduced analogue of a thermodynamic entropy balance: mechanical energy is dissipated and converted into entropy at a rate encoded geometrically by the Herglotz Lagrangian.

Remark 8 *(Comparison with the heavy top in a Stokes flow)* The thermoviscous rigid body considered above is closely related to the thermomechanical rigid body models of Couéraud and Gay–Balmaz, notably the heavy top in a Stokes flow (Couéraud and Gay-Balmaz 2020). There, the Lagrangian consists of a mechanical part and an internal energy $U_B(S)$ depending only on the entropy, while viscous effects enter through phenomenological constraints that produce a torque of Stokes type and an associated entropy production law. The Lagrangian itself encodes only the reversible part of the dynamics.

In contrast, the Herglotz formulation used here incorporates a simple Rayleigh dissipation potential directly into the variational principle, through the term $\frac{\gamma}{2T_0}\|\xi\|^2$ in $L(R, \xi, S)$. The internal contribution is still purely entropic (here $T_0 S$, the isothermal case of the general $U_B(S)$), but the velocity-dependent dissipation is now encoded geometrically via the Herglotz term. As a result, the Euler–Poincaré–Herglotz equation contains a multiplicative damping term $-T_0\Pi(\xi)$, and the associated Herglotz energy decays exponentially rather than satisfying a conservation law for the sum of mechanical and internal energies.

Thus the present model provides a fully variational, symmetry-reduced counterpart of the thermomechanical heavy-top dynamics, with dissipation implemented intrinsically through the Herglotz formalism rather than by external constraints. ◇

7.3 Hamilton–Pontryagin–Herglotz Principle

Example 8 *(Geodesic flow with Herglotz-type dissipation)* Let (Q, g) be a Riemannian manifold and take $E = TQ$ with its standard Lie algebroid structure ($\rho = \mathrm{id}$ and bracket the Lie bracket of vector fields). We will denote by ∇^g the Levi–Civita connection of g and by ∇ a locally trivial connection used to derived the local expression of the intrinsic equations as in the proof of Lemma 1. Consider the Herglotz Lagrangian

$$L(q, \dot{q}, z) = \tfrac{1}{2}\, g_q(\dot{q}, \dot{q}) - V(q) - \gamma z, \qquad \gamma > 0.$$

Then, using the connection ∇ we obtain

$$p_q = dL_{\mathrm{ver}}(q, \dot{q}, z) = g_q(\dot{q}, \cdot), \qquad dL_{\mathrm{hor}}(q, \dot{q}, z) = -dV(q), \qquad \frac{\partial L}{\partial z} = -\gamma.$$

And the intrinsic Euler–Lagrange–Poincaré–Herglotz equation becomes

$$\dot{p}_q = -\,dV(q) - \gamma\, p_q.$$

Using the musical isomorphism $\sharp : T^*Q \to TQ$ defined by $g(\sharp(p_q), v_q) = \langle p_q, v_q\rangle$ for any $v_q \in TQ$, we may conclude that $\sharp(p_q) = \dot{q}$, and using that $\sharp(\dot{p}_q) = \nabla^g_{\dot{q}}\dot{q}$

yields the damped Newton–Lagrange equation

$$\nabla^g_{\dot q}\dot q = -\mathrm{grad}_g V(q) - \gamma\,\dot q,$$

where $\mathrm{grad}_g V = \sharp(dV)$. Thus a Riemannian geodesic with potential acquires a natural linear damping term generated by the Herglotz variable, reproducing Rayleigh friction in a completely coordinate-free way.

Example 9 *(Euler–Poincaré–Herglotz equations)* Take the Lie algebroid $E = \mathfrak{g}$ with anchor 0 and bracket the Lie bracket on $\mathfrak{g}$. Since the base is a point, there is no horizontal part. If we start with the trivial E–connection $\nabla \equiv 0$, then $\overline{\nabla}$ is simply the adjoint representation:

$$\overline{\nabla}_\xi X = [\xi, X] = \mathrm{ad}_\xi\,X, \qquad \overline{\nabla}^*_\xi \mu = -\,\mathrm{ad}^*_\xi\,\mu.$$

Let $\ell : \mathfrak{g}\times\mathbb{R} \to \mathbb{R}$ be a reduced Herglotz Lagrangian, with reduced velocity $\xi(t) \in \mathfrak{g}$ and reduced momentum $\mu = \partial\ell/\partial\xi(\xi, z)$. The Herglotz equation is $\dot z = \ell(\xi, z)$.

Applying Theorem 2 gives the *Euler–Poincaré–Herglotz equation*

$$\dot\mu = \mathrm{ad}^*_\xi\,\mu + \frac{\partial\ell}{\partial z}(\xi, z)\,\mu,$$

a dissipative deformation of the classical Euler–Poincaré equation (Anahory et al. 2024).

Example 10 *(Charged particle in a magnetic field on an Atiyah algebroid)*

Let $\pi : Q \to M$ be a principal S^1–bundle endowed with a principal connection $\mathcal{A} : TQ \to \mathfrak{g}$ and curvature $B : TQ \times TQ \to \mathfrak{g}$, representing a magnetic field on M. The associated Atiyah algebroid is $E = TQ/S^1 \to M$. Using the connection $\mathcal{A}$, one has the standard Lie algebroid identification

$$E \cong TM \oplus (Q\times\mathbb{R})/S^1,$$

where $\tilde{\mathfrak{g}} := (Q\times\mathbb{R})/S^1$ is the adjoint bundle over M. A typical element $a \in E_x$ is written as $a = (X, \bar u)$, where $X \in T_xM$ represents the horizontal velocity and $\bar u = [(q, u)]$ with $u \in \mathbb{R}$ and $\pi(q) = x$ is the internal charge variable. The projection is simply $\tau(X, \bar u) = \tau_M(X)$, where $\tau_M : TM \to M$ is the tangent bundle projection, the anchor is the projection onto the first component $\rho(X, \bar u) = X$ and the bracket is given by

$$[(X, \bar u), (Y, \bar v)] = \Big([X, Y], \tilde\nabla_X\bar v - \tilde\nabla_Y\bar u - \tilde B(X, Y)\Big),$$

where $\tilde\nabla$ is the associated connection on $\tilde{\mathfrak{g}}$ to the E-connection ∇ derived from the principal connection $\mathcal{A}$ and $\tilde B : TM \times TM \to \mathfrak{g}$ is the reduced principal connection curvature defined by $\tilde B(X, Y) = [(q, B(X^h, Y^h))]$, for $X, Y \in T_xM$ and X^h denotes the horizontal lift (see Cendra et al. 2001 for more details).

Let g be a Riemannian metric on M, $m > 0$ and $e \in \mathbb{R}$ be the mass and electric charge of the particle and $V : M \to \mathbb{R}$ a potential function defined on M. Consider the Herglotz Lagrangian

$$L(x, \dot{x}, u, z) = \tfrac{m}{2}\, g_x(\dot{x}, \dot{x}) + e\, u - V(x) - \gamma z, \qquad \gamma > 0.$$

With respect to a TM–connection ∇ on E of the form

$$\nabla_X(Y, \bar{u}) = (\nabla^M_X Y, \tilde{\nabla}_X \bar{u}),$$

where $\nabla^M_X Y$ is a locally trivial affine connection on M and $\tilde{\nabla}$ is the associated connection on $\tilde{\mathfrak{g}}$ defined previously, the vertical differential of L is

$$dL_{\mathrm{ver}}(x, \dot{x}, u, z) = (m\, g_x(\dot{x}, \cdot),\ e) \in T^*_x M \oplus \mathbb{R}^*,$$

which defines the momentum

$$p = (p_x, p_u) = (m\, g(\dot{x}, \cdot),\ e).$$

The horizontal differential and partial derivative with respect to z are

$$dL_{\mathrm{hor}} = -dV, \qquad \frac{\partial L}{\partial z} = -\gamma.$$

In addition, the connection $\overline{\nabla}$ on E satisfies

$$\overline{\nabla}_{(X,\bar{u})}(Y, \bar{v}) = \nabla_{(X,\bar{u})}(Y, \bar{v}) - (0, \tilde{B}(X, Y)).$$

Applying the intrinsic Euler–Lagrange–Poincaré–Herglotz equation (19) yields

$$\overline{\nabla}^*_{(\dot{x},\bar{u})} p = -\rho^*(dV) - \gamma p.$$

Given a time-dependent section of E^* such as $p(t) = \mu(t, x(t))$, we have that

$$\begin{aligned}\langle \overline{\nabla}^*_{(\dot{x},\bar{u})} p(t), (Y, \bar{v})\rangle &= \langle \dot{p}, (Y, \bar{v})\rangle - \langle p(t), \overline{\nabla}_{(\dot{x},\bar{u})}(Y, \bar{v})\rangle \\ &= \langle \dot{p}, (Y, \bar{v})\rangle - \langle p(t), \nabla_Y(\dot{x}, \bar{u}) + [(\dot{x}, \bar{u}), (Y, \bar{v})]\rangle\end{aligned}$$

which gives

$$\langle \overline{\nabla}^*_{(\dot{x},\bar{u})} p(t), (Y, \bar{v})\rangle = \langle \dot{p}, (Y, \bar{v})\rangle - \langle p, (\overline{\nabla}^M_X Y, \tilde{\nabla}_X \bar{v} - \tilde{B}(X, Y))\rangle.$$

Projecting onto the component along TM and using that the connection $\overline{\nabla}^M$ is locally trivial, we obtain a coordinate expression of the form

$$\dot{p}_x + \tilde{B}(\dot{x}, \cdot) p_u = -dV - \gamma p_x.$$

where $(0, \tilde{B}(\dot{x}, \cdot))^*$ is an adjoint operator defined by $\langle (0, \tilde{B}(\dot{x}, \cdot))^* p, Y\rangle = \langle p, (0, \tilde{B}(\dot{x}, Y))\rangle$. Using the musical isomorphism $\sharp : T^*M \to TM$, we obtain the *magnetic geodesic equation with Herglotz damping*

$$m\, \nabla^g_{\dot{x}} \dot{x} = -e\, \sharp(\iota_{\dot{x}} \tilde{B}) - \mathrm{grad}_g V(x) - \gamma m\, \dot{x}, \qquad \dot{z} = L(x, \dot{x}, u, z).$$

where $\mathrm{grad}_g V = \sharp(dV)$, $(\iota_{\dot{x}} \tilde{B})(Y) = \tilde{B}(\dot{x}, Y)$ and ∇^g is the Levi–Civita connection of (M, g).

Thus the curvature of the Atiyah algebroid reproduces the magnetic Lorentz force, while the Herglotz term produces a linear dissipation of kinetic energy in a fully intrinsic and gauge-invariant manner.

8 Conclusions and Future Work

In this paper, we developed a variational framework for *dissipative mechanics on Lie algebroids*. We derived the Euler–Lagrange–Herglotz equations in local coordinates and then obtained a fully coordinate-free, connection-based formulation of the Euler–Lagrange–Poincaré–Herglotz equations, where the connection is used only as an auxiliary device for the horizontal-vertical splitting. This unifies classical contact mechanics, Euler-Poincaré and Lagrange-Poincaré reduction with linear dissipation.

We also introduced the Hamilton–Pontryagin–Herglotz principle on a Lie algebroid, yielding an implicit system that captures simultaneously kinematic constraints, the momentum relation, the horizontal dynamics, and the Herglotz evolution equation. This formulation is well adapted to generalizations to discrete geometry and to the systematic treatment of symmetries, reduction, and curvature effects. Moreover, our framework leads naturally to energy balance laws and Noether–Herglotz theorems, showing how classical conserved quantities are replaced by *dissipated invariants* determined by an integrating factor.

A natural continuation of the present work is to explore deeper intrinsic geometric approaches to Herglotz variational framework based on formulations of dissipative mechanics on algebroids, such as Tulczyjew-type contact approaches (Grabowska and Grabowski 2024), intrinsic variational formalisms (Grabowska and Grabowski 2008), and optimal-control formalisms (Grabowski and Jóźwikowski 2011; Colombo 2025).

The geometric nature of our results suggests several important research directions. A central objective is the construction of *geometric structure-preserving contact integrators* on *Lie groupoids*, obtained by discretizing either Herglotz variational principle or the Hamilton–Pontryagin–Herglotz principle. Contact variational integrators for the standard phase space $TQ \times \mathbb{R}$ were studied in Vermeeren et al. (2019) and Simoes et al. (2021) but the analogue of contact variational integrators on groupoids remains unexplored. Such integrators would preserve admissibility at the groupoid level, reproduce the correct exponential energy decay, and retain the dissipated Noether quantities exactly .

A Appendix: Local Expression of the Intrinsic Herglotz Equation

Lemma 1 *Let $\{e_\alpha\}$ be a local frame of E over a coordinate chart (x^i) on M, with*

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma, \qquad \rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}.$$

Let ∇ be a TM-connection on E, and write its local coefficients as

$$\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma^\gamma_{i\alpha} e_\gamma.$$

Let

$$a(t) = y^\alpha(t) e_\alpha|_{x(t)}, \qquad p(t) = p_\alpha(t) e^\alpha|_{x(t)},$$

where $\{e^\alpha\}$ is the dual frame, and let

$$p = dL^{\mathrm{ver}}(a, z).$$

Then the intrinsic Euler–Lagrange–Poincaré–Herglotz equation

$$\nabla^*_a p(t) - \rho^*\big(dL^{\mathrm{hor}}(a(t), z(t))\big) - \frac{\partial L}{\partial z}(a(t), z(t))\, p(t) = 0$$

takes the local form

$$\dot{p}_\alpha + C^\gamma_{\alpha\beta}(x)\, y^\beta\, p_\gamma - \rho^i_\alpha(x) \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial z}(x, y, z)\, p_\alpha = 0,$$

which is exactly the Euler–Lagrange–Herglotz system of Theorem 1.

Proof Choosing the TM-connection on E with local Christoffel symbols determined by $\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma^\gamma_{i\alpha} e_\gamma$, the vertical and horizontal components of dL are

$$dL_{\mathrm{ver}} = \frac{\partial L}{\partial y^\alpha} e^\alpha, \quad dL_{\mathrm{hor}} = \left(\frac{\partial L}{\partial x^i} - \Gamma^\gamma_{i\beta} y^\beta \frac{\partial L}{\partial y^\gamma}\right) dx^i,$$

which may be easily derived using the vertical lift $V : E \to TE$ and the horizontal lift $H : TM \to TE$ determined by the connection, whose local expressions are $V(e_\alpha) = \frac{\partial}{\partial y^\alpha}$ and $H(\partial/\partial x^i) = \frac{\partial}{\partial x^i} - \Gamma^\gamma_{i\beta} y^\beta \frac{\partial}{\partial y^\gamma}$.

In addition, given a time-dependent section of E^* such as $p(t) = \mu(t, x(t))$, we have that

$$\begin{aligned}\langle \overline{\nabla}^*_a p(t), e_\alpha\rangle &= \dot{p}_\alpha - \langle p(t), \overline{\nabla}_a e_\alpha\rangle = \dot{p}_\alpha - y^\beta \langle p(t), \nabla_{\rho(e_\alpha)} e_\beta + [e_\beta, e_\alpha]\rangle \\ &= \dot{p}_\alpha - (\rho^i_\alpha y^\beta \Gamma^\gamma_{i\beta} + y^\beta C^\gamma_{\beta\alpha}) p_\gamma = \dot{p}_\alpha + y^\beta C^\gamma_{\alpha\beta} p_\gamma - \rho^i_\alpha y^\beta \Gamma^\gamma_{i\beta} p_\gamma.\end{aligned}$$

On the other hand,

$$\langle \rho^*(dL_{\text{hor}}), e_\alpha \rangle = \langle dL_{\text{hor}}, \rho(e_\alpha) \rangle = \rho^i_\alpha \left(\frac{\partial L}{\partial x^i} - \Gamma^\gamma_{i\beta} y^\beta \frac{\partial L}{\partial y^\gamma} \right).$$

Substituting these expressions into

$$\nabla^*_a p - \rho^*(dL^{\text{hor}}) - \frac{\partial L}{\partial z} p = 0,$$

using $p_\gamma = \partial L / \partial y^\gamma$ and pairing with e_α, we obtain

$$\dot{p}_\alpha + C^\gamma_{\alpha\beta} y^\beta p_\gamma - \rho^i_\alpha \Gamma^\gamma_{i\beta} y^\beta p_\gamma - \rho^i_\alpha \frac{\partial L}{\partial x^i} + \rho^i_\alpha \Gamma^\gamma_{i\beta} y^\beta p_\gamma - \frac{\partial L}{\partial z} p_\alpha = 0.$$

The terms containing the connection coefficients $\Gamma^\gamma_{i\beta}$ cancel, and we are left with

$$\dot{p}_\alpha + C^\gamma_{\alpha\beta} y^\beta p_\gamma - \rho^i_\alpha \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial z} p_\alpha = 0.$$

Finally, using again $p_\alpha = \partial L / \partial y^\alpha$, we recover

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha} \right) + C^\gamma_{\alpha\beta}(x) y^\beta \frac{\partial L}{\partial y^\gamma} - \rho^i_\alpha(x) \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial z} \frac{\partial L}{\partial y^\alpha} = 0,$$

which is exactly the Euler–Lagrange–Herglotz equation of Theorem 1.

□

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Data Availability No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare no conflict of interest.

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