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Mechanical Hamiltonization of Unreduced ϕ-Simple Chaplygin Systems

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ABSTRACT

In this paper, we prove that the trajectories of unreduced ϕ-simple Chaplygin kinetic systems are reparameterizations of horizontal geodesics with respect to a modified Riemannian metric. Furthermore, our proof is constructive and these Riemannian metrics, which are not unique, are obtained explicitly in interesting examples. We also extend these results to ϕ-simple Chaplygin mechanical systems (not necessarily kinetic).

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1 | Introduction

1.1 | Hamiltonization, Our Problem, and Invariant Volume Forms of Reduced ϕ-Simple Chaplygin Systems

A classical theory in nonholonomic mechanics is the so-called Hamiltonization problem (cf. [1–4]). This problem, which has received a lot of attention in recent years (see [5] and the references therein; see also [6]), consists in arguing when a symmetric nonholonomic mechanical system admits a Hamiltonian formulation, up to reparameterization, after reduction by symmetries.

Now, suppose that a nonholonomic mechanical system with symmetry is Hamiltonizable. Then, a natural question arises: *does the unreduced system also admit a Hamiltonian formulation?* Since we deal exclusively with nonholonomic mechanical systems, the previous question may be even more specialized: *are there a (modified) Riemannian metric and a (modified) potential energy such that the trajectories of the new unconstrained mechanical system, with initial velocity in the constraint distribution, are reparameterizations of the nonholonomic trajectories of the unreduced nonholonomic mechanical system?*

This question is a particular case of a more general problem, which was posed in [7]: the kinetic Lagrangianization of kinetic nonholonomic systems or, more generally, the formulation of a nonholonomic mechanical system as an unconstrained Lagrangian mechanical system (see items 3 and 7 of Section 6 in [7]). We will provide an answer to this question for a class of nonholonomic systems, called ϕ-simple Chaplygin systems, introduced in a geometric way in [6] (see also [8] for a local discussion).

ϕ-Simple Chaplygin systems are Hamiltonizable after reduction by symmetries. We recall that a Chaplygin system is a nonholonomic mechanical system whose constraint distribution is the horizontal subbundle associated with a principal connection in a principal G-bundle and, in addition, the Lie group G is a symmetry for the system (see [9]; see also [10, 11]). A tensor field $\mathcal{T}$ of type $(1, 2)$ on the base space $\bar{Q} = Q/G$ of the principal bundle, called the gyroscopic tensor, plays an important role in the description of the geometry of the reduced nonholonomic system. $\mathcal{T}$ measures the interplay between the kinetic Riemannian metric and the nonintegrability of the constraint distribution. In fact, using $\mathcal{T}$, one may construct an almost symplectic structure Ω_{nh} on the reduced phase space $T^*(Q/G)$ such that the reduced nonholonomic dynamics is Hamiltonian with respect to Ω_{nh}. If the gyroscopic tensor $\mathcal{T}$ satisfies

$$\mathcal{T} = Id \otimes d\phi - d\phi \otimes Id \tag{1}$$

with $\phi \in C^\infty(\bar{Q})$, the Chaplygin system is ϕ-simple [6].

A local characterization of ϕ-simple Chaplygin systems and nontrivial examples of such systems were obtained in [8] (see also [6, 12]). Moreover, in [6], it was proved that a Chaplygin system is ϕ-simple if and only if the almost symplectic structure Ω_{nh} is conformally symplectic (we stress that the proof of the direct implication of this statement is a consequence of a result by Stanchenko [13]). So, as we mentioned before, ϕ-simple Chaplygin systems are Hamiltonizable after reduction. In fact, the conformal factor for Ω_{nh} is related with the invariant volume form for the reduction of the Chaplygin system (for more details, see [6]). Thus, a necessary condition for a Chaplygin system to be ϕ-simple is the existence of an invariant volume form for the reduced system. In this direction, we remark that this technique originated in Chaplygin's work [1] (see also [14]) on integrability of nonholonomic systems and relies on the presence of an integral invariant, which is not always the case. Also, invariant volume forms for the reduction of Chaplygin systems were discussed in [15], for general nonholonomic mechanical systems in [16] (see also [17]) and for nonholonomic systems on Lie groups in [18]. In addition, very recently, ϕ-simple Chaplygin systems with gyroscopic terms were considered in [19].

1.2 | Purpose of the Paper, Related Previous Contributions, and Our Strategy

The contribution of our paper is to give a positive answer to the questions in the beginning of the previous section for ϕ-simple Chaplygin systems.

We remark that, comparatively speaking, the Hamiltonization of unreduced nonholonomic mechanical systems has received much less attention than the Hamiltonization problem for the reduction of symmetric nonholonomic mechanical systems. In fact, to our knowledge, only a few papers (see [20, 21] and also [22]) have discussed this problem.

In these papers, and particularly in [21], the idea is to use the inverse problem of variational calculus to provide a set of conditions for the existence of a Riemannian metric, the geodesic extension in the terminology of [21], whose geodesics with initial velocity in the constraint distribution are the trajectories of the kinetic nonholonomic system.

Our approach to the problem is different. In fact, we will use the symmetries of the ϕ-simple Chaplygin kinetic system and the Riemannian submersion theory ([23–25]) to obtain a modified Riemannian metric such that the nonholonomic trajectories are reparameterizations of geodesics of the new metric. Our method is constructive and we can give an explicit expression of the new metric. Moreover, using the previous construction for ϕ-simple Chaplygin kinetic systems, we obtain the corresponding results for the more general case of a ϕ-simple Chaplygin mechanical system (adding a potential energy).

In the particular case of a ϕ-simple Chaplygin kinetic system, the Riemannian submersion theory is used as follows. For a Riemannian submersion, the horizontal distribution Hor in the total space is just the orthogonal subbundle of the vertical subbundle. Hor is, in general, nonintegrable and moreover, the restriction of the geodesic flow to Hor is tangent. This means that if the initial velocity of a geodesic is horizontal then its velocity curve will be entirely contained in Hor. In addition, the projection of a horizontal geodesic is a geodesic in the base space for the reduced metric and the horizontal lift of a geodesic in the base space is a horizontal geodesic in the total space (for more details, see [23–25]). Based on this fact, and for a ϕ-simple Chaplygin kinetic system, our goal in the paper will be to reproduce this picture by constructing a new metric with respect to which the vertical space is orthogonal to the constraint distribution. In this scenario, the horizontal geodesics coincide with the nonholonomic trajectories. The relevant mechanism is the existence of a projection mapping nonholonomic trajectories to geodesics or, at least, onto reparameterizations of geodesics. In this case, we are able to tweak the original metric in order to make the horizontal space orthogonal to the vertical space and still preserve the Riemannian submersion structure.

1.3 | Results of the Paper

For a Chaplygin mechanical system, we will use the following notation:

$$(Q, g, V, G, \mathcal{D}),$$

where Q is the configuration space, g is a G-invariant Riemannian metric on Q, $V : Q \to \mathbb{R} \in C^\infty(Q)$ is the G-invariant potential energy, G is the symmetry Lie group, and $\mathcal{D} \subseteq TQ$ is the constraint distribution. When $V = 0$, the system is kinetic. The base space of the principal G-bundle is $\bar{Q} = Q/G$ and the Riemannian metric $\bar{g}$ on $\bar{Q}$ is characterized by the condition

$$\bar{g}(T\pi(X), T\pi(Y)) = g(X, Y), \text{ for } X, Y \in \mathcal{D},$$

with $\pi : Q \to \bar{Q}$ the principal bundle projection.

We will first consider in detail the characterization of trajectories of ϕ-simple kinetic Chaplygin systems as reparameterizations of geodesics with respect to a modified metric h (Theorem 3.4), which is not unique but may be chosen as follows:

1. $h(X, Y) = \exp(2\phi \circ \pi) g(X, Y), \forall X, Y \in \Gamma(\mathcal{D})$;
2. $h(Z, W) = g(Z, W), \forall Z, W \in \Gamma(V\pi)$;
3. $h(X, Z) = 0, \forall X \in \Gamma(\mathcal{D}),\ Z \in \Gamma(V\pi)$.

Here, $V\pi$ is the vertical bundle associated to π. Such metric will be called the *principal Riemannian metric*. However, the main result of the paper concerns the general case of ϕ-simple Chaplygin systems where one has a G-invariant potential function in addition to the kinetic energy.

Theorem 4.3. *Let $(Q, g, V, G, \mathcal{D})$ be a Chaplygin system with G-invariant potential V. If the system is ϕ-simple, there exists a Riemannian metric h on Q such that all the mechanical trajectories associated with h and V with initial velocity in $\mathcal{D}$ are reparameterizations of the nonholonomic trajectories associated with the Chaplygin system. In fact, h may be chosen as the principal Riemannian metric.*

Note that our result is constructive, since it provides an explicit construction of the principal Riemannian metric, despite the fact that this is not the only possible choice of Riemannian metric h.

An immediate consequence of our main theorem is an interesting result. Namely, the nonholonomic trajectories for a kinetic ϕ-simple Chaplygin system become locally minimizing curves with respect to the new metric, as the following corollary states.

Corollary 3.7. *Let $(Q, g, G, \mathcal{D})$ be a kinetic Chaplygin system. Suppose that the system is ϕ-simple and h is the principal Riemannian metric. If $c : [0, T] \to Q$ is a nonholonomic trajectory there exists $0 < t_0 < T$ such that for all $t < t_0$ the length of the curve $c : [0, t] \to Q$ is just the Riemannian distance associated with h between $c(0)$ and $c(t)$.*

As a demonstration of the applicability in practical examples, we have constructed explicit examples of Riemannian metrics whose geodesics with initial velocity satisfying the nonholonomic constraints coincide, up to reparameterization, with the trajectories of the initial nonholonomic system.

For instance, in the classical example of the vertical rolling disk, the principal Riemannian metric is

$$h = mdx^2 + mdy^2 + (I + 2mR^2)d\theta^2 + Jd\varphi^2$$
$$- mR\cos\varphi\, dxd\theta - mR\sin\varphi\, dyd\theta.$$

Here, (x, y, θ, φ) are standard coordinates in the configuration space $Q = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, m and R are the mass and radius of the disk, and I, J are the inertia.

In fact, in Example 1 of Section 5, we compute the geodesic equations for h and deduce that the geodesics with initial velocity in the constraint distribution coincide with the nonholonomic trajectories, without the need for a reparameterization.

In the nonholonomic particle (see Example 2 in Section 5), the configuration space is $\mathbb{R}^3$, (x, y, z) are the standard coordinates on $\mathbb{R}^3$ and our method allows us to construct the principal Riemannian metric

$$h = (1 + y^2)dx^2 + \frac{dy^2}{1 + y^2} + dz^2 - ydxdz,$$

whose geodesics with initial velocity satisfying $\dot{z} = y\dot{x}$ are precisely the nonholonomic trajectories, up to a reparameterization.

Another interesting example is the case of the Veselova problem where we also provide (see Example 3 in Section 5) the principal Riemannian metric describing the nonholonomic trajectories, which in right-trivialized coordinates reads $h((R, \omega), (R, \omega)) = (H\omega, \omega)$, with

$$H = \frac{1}{(\mathbb{I}^{-1}\gamma, \gamma)}\left(I - e_3 e_3^T\right)^\top R\mathbb{I}R^{-1}\left(I - e_3 e_3^T\right) + \left(e_3 e_3^T\right)^\top R\mathbb{I}R^{-1}\left(e_3 e_3^T\right).$$

Here, $Q = SO(3)$ is the configuration space of the system, $R \in SO(3)$, $\mathbb{I}$ is the inertia tensor, $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$, $\gamma = R^T e_3 \in \mathbb{S}^2 = SO(3)/SO(2)$, and I is the identity matrix.

In all three cases, we can observe that the reparameterization is related to the function ϕ, determining the ϕ-simple nature of the system. The vertical rolling disk is a special case where $\phi \equiv 0$ forcing the associated reparameterization to be the identity.

We remark that the method has been applied to the three previous examples but, since it is constructive, it may be applied to more complex cases (see Remark 5.2).

1.4 | Organization of the Paper

The paper is organized as follows. In Section 2, we first briefly review kinetic nonholonomic systems, the notion of a principal connection on a principal G-bundle, Riemannian submersions, and Chaplygin systems. This section is intended to make the paper self-contained and settle the notation that we use throughout the paper. In Section 3, we prove a first important result (Theorem 3.4): unreduced ϕ-simple Chaplygin kinetic systems are Hamiltonizable, up to reparameterization, by geodesics. In Section 4, we show that the previous results are not essentially changed by including a G-invariant potential function in the picture. This is the main result of the paper (Theorem 4.3). In Section 5, we present examples of typical ϕ-simple Chaplygin mechanical systems (the vertical rolling disk, the nonholonomic particle, and the Veselova system) where we construct the principal Riemannian metric Hamiltonizing the nonholonomic trajectories. Finally, in Section 6, we present some lines of research for future work.

2 | Preliminaries

2.1 | Nonholonomic Systems

Throughout this section, suppose that (Q, g) is a Riemannian manifold, $\mathcal{D}$ is a nonintegrable distribution on Q and consider the *kinetic nonholonomic system* defined by the Lagrangian function $L_g : TQ \to \mathbb{R}$ given by

$$L_g(v_q) = \frac{1}{2}g(v_q, v_q), \quad v_q \in T_qQ$$

and also by the nonintegrable distribution $\mathcal{D}$. We may define two projections resulting from the decomposition of the tangent bundle induced by the metric g

$$TQ = \mathcal{D} \oplus \mathcal{D}^\perp,$$

that is, the projection to $\mathcal{D}$ and to $\mathcal{D}^{\perp}$ given by $\mathcal{P} : TQ \to \mathcal{D}$ and $\mathcal{Q} : TQ \to \mathcal{D}^{\perp}$, respectively.

The trajectories of the nonholonomic system can be described as geodesics but relative to a nonsymmetric connection (see [26])

$$\nabla^{nh}_X Y = \nabla^g_X Y + (\nabla_X \mathcal{Q})Y, \quad X, Y \in \mathfrak{X}(Q),$$

where ∇^g is the Levi-Civita connection. Therefore, a curve $c : [0, h] \to Q$ is a trajectory of the kinetic nonholonomic system $(L_g, \mathcal{D})$ if and only if it satisfies the equations

$$\nabla^{nh}_{\dot{c}(t)} \dot{c}(t) = 0, \quad \dot{c}(0) \in \mathcal{D}. \tag{2}$$

Remark 2.1. Equation (2) is equivalent to the Lagrange–d'Alembert equations associated with the Lagrangian function L_g ([10, 27, 28]). These equations are given by

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}^i}\right) - \frac{\partial L_g}{\partial q^i} = \lambda_a \mu^a_i(q) \\ &\mu^a_i(q)\dot{q}^i = 0, \end{aligned} \tag{3}$$

where (q^i) are local coordinates on Q, $(q^i, \dot{q}^i)$ are the corresponding local coordinates on TQ, $\mu^a = \mu^a_i dq^i$ are local 1-forms on Q such that $\mathcal{D} = \{v_q \in T_qQ \mid \mu^a(v_q) = 0)\}$, and λ_a are Lagrange multipliers that can be determined considering the total derivative of the constraint equations $\mu^a_i(q)\dot{q}^i = 0$.

Throughout the paper, we will denote by Γ_g the geodesic vector field, that is, the vector field on TQ whose trajectories are the geodesics of g; and by $\Gamma_{(g,\mathcal{D})}$ the vector field on $\mathcal{D}$ whose trajectories are nonholonomic trajectories, that is, satisfy Equation (2).

2.2 | Principal Fiber Bundles

We include here a basic treatment of principal fiber bundles to make the paper more self-contained and fix some notation. The interested reader can read more about the subject in [29–31], for instance.

Let $\Phi : G \times Q \to Q$ be a *left action* of a Lie group G on a smooth manifold Q, denoted by $\Phi(g, q) = g \cdot q = \Phi_g(q) = \Phi_q(g)$. The orbit of the action through a point $q \in Q$ is the set $\mathrm{Orb}(q) = \{g \cdot q \mid g \in G\}$. Denote by $\mathfrak{g}$ the Lie algebra of the group G. For each element ξ in the Lie algebra there is a vector field on Q called the *infinitesimal generator* of the group action denoted by ξ_Q and defined by $\xi_Q(q) = T_e\Phi_q(\xi)$ where e is the identity element of G. If we assume that the action Φ is free and proper we can endow the quotient space $\bar{Q} = Q/G$ with a manifold structure under which the natural projection $\pi : Q \to \bar{Q}$ is a surjective submersion. Therefore, associated to the left action Φ we have a fiber bundle π satisfying the following properties:

1. each π-fiber, denoted by $Q_{[q]} = \pi^{-1}([q])$, is an orbit of the action;
2. the standard fiber is G;
3. the local trivialization $\{U, \psi\}$ of the fiber bundle is *equivariant*, that is, given $U \subseteq \bar{Q}$, an open subset of $\bar{Q}$, the trivialization

$$\begin{aligned} \psi : \pi^{-1}(U) &\to U \times G \\ q &\mapsto (\pi(q), \psi_2(q)) \end{aligned}$$

satisfies $\psi_2(g \cdot q) = g \cdot \psi_2(q)$, where we are considering the group multiplication on the right-hand side.

The above properties define the *principal G-bundle* $(Q, \bar{Q}, G, \pi)$, where Q is the *bundle space*, $\bar{Q}$ is the *base space*, G is the *structure group*, and π is the *projection*. The *vertical space* at points $q \in Q$, denoted by $V_q\pi$, form a distribution $V\pi$ on Q and defined as the kernel of $\pi_* \equiv T\pi : TQ \to T\bar{Q}$. The vectors contained in $V\pi$ are called *vertical vectors*. Notice that vertical vectors are just tangent vectors to the orbits of G. Explicitly,

$$V_q\pi = T_q(\mathrm{Orb}(q)) = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}.$$

A *principal connection* on the principal G-bundle is a smooth distribution $\mathcal{H}$ on Q satisfying the following properties:

1. $T_qQ = V_q\pi \oplus \mathcal{H}_q$, for every $q \in Q$;
2. The distribution is G-invariant, that is, $\mathcal{H}_{g\cdot q} = (\Phi_g)_*(\mathcal{H}_q)$.

$\mathcal{H}$ is called the *horizontal distribution* determined by the connection and the vectors contained in $\mathcal{H}$ are called *horizontal vectors*. Many authors give an alternative equivalent definition of a principal connection as a $\mathfrak{g}$-valued one-form $\omega : TQ \to \mathfrak{g}$ such that

1. $\omega(\xi_Q(q)) = \xi$ for every $\xi \in \mathfrak{g}$;
2. $\omega((T_q\Phi_g)(X)) = \mathrm{Ad}_g(\omega(X))$ for every $X \in T_qQ$.

Recall that the adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the push-forward at the identity of the conjugation map $C_g : G \to G, h \mapsto ghg^{-1}$. Such a map $\omega : TQ \to \mathfrak{g}$ is called the *connection form* determined by the principal connection. Here, we may recover the previous properties by defining the *horizontal subspace* at q as $\mathcal{H}_q = \ker \omega|_{T_qQ}$. Conversely, given a horizontal distribution $\mathcal{H}$, we may prove that a $\mathfrak{g}$-valued one-form ω satisfying $\omega(X) = 0$ for every $X \in \mathcal{H}$ and $\omega(\xi_Q) = \xi$ for every $\xi \in \mathfrak{g}$ is a connection form. The only nonobvious fact needed to prove this is the equality

$$(T_q\Phi_g)(\xi_Q(q)) = [\mathrm{Ad}_g(\xi)]_Q(g \cdot q).$$

Given a principal connection, every vector $X \in T_qQ$ can be uniquely written as

$$X = \mathrm{hor}(X) + \mathrm{ver}(X),$$

where $\mathrm{hor} : TQ \to \mathcal{H}$ and $\mathrm{ver} : TQ \to V\pi$ are, respectively, the *horizontal* and the *vertical projectors* associated to the decomposition of the tangent space determined by the connection.

The *horizontal lift* of a vector field $X \in \mathfrak{X}(\bar{Q})$ on the base space is the unique horizontal vector field $X^h \in \mathfrak{X}(Q)$ on the bundle space that projects onto X, that is, $(T_q\pi)(X^h(q)) = X(\pi(q))$.

A vector field $X \in \mathfrak{X}(Q)$ on the bundle space is *G-invariant* if $X(g \cdot q) = (T_q\Phi_g)(X(q))$. For each G-invariant vector field X on Q, there exists a unique *reduced vector field* $\overline{X} \in \mathfrak{X}(\bar{Q})$ on the base space such that the following diagram is commutative:

$$\begin{array}{ccc} TQ & \xrightarrow{T\pi} & T\bar{Q} \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle \overline{X}} \\ Q & \xrightarrow{\pi} & \bar{Q} \end{array}$$

Moreover, integral curves of X project to integral curves of $\overline{X}$. On the other hand, in the setting of Marsden–Weinstein symplectic reduction, a theoretical reconstruction process has been proposed to obtain the integral curves of the unreduced Hamiltonian vector field X from the integral curves of the reduced Hamiltonian vector field $\overline{X}$. In the particular case when the symmetry Lie group G is abelian this is easy. But when G is not abelian, the explicit integration of $\overline{X}$ may be a hard task (see [29, 32]).

Definition 2.2. A horizontal lift of a curve $\bar{c} : I \to \bar{Q}$ on $\bar{Q}$ is a curve $c : I \to Q$ on Q such that $\pi(c(t)) = \bar{c}(t)$ for all $t \in I$ and such that $\dot{c}(t) \in \mathcal{H}_{c(t)}$ for all $t \in I$, where $\mathcal{H}$ is the horizontal distribution associated to a given principal connection.

In fact, if $q_0 \in Q$ and $\bar{c}(t_0) = \pi(q_0)$ then there exists a unique horizontal lift $c : I \to Q$ of $\bar{c} : I \to \bar{Q}$ such that $c(t_0) = q_0$.

Lemma 2.3. *If a curve c_1 in $\bar{Q}$ is a reparameterization of a curve c_2 in $\bar{Q}$, then the horizontal lift of c_1 to Q is a reparameterization of the horizontal lift of the curve c_2 to Q.*

Proof. Let $c_1 : I \to \bar{Q}$, $\varphi : J \to I$ a reparameterization and $c_2 : J \to \bar{Q}$ defined by $c_2(t) = (c_1 \circ \varphi)(t)$ with $I = [s_0, s_1]$ and $J = [t_0, t_1]$. Denote by $c_1^h(t)$ the horizontal lift of the curve c_1 satisfying $c_1^h(s_0) = q \in Q$. The curve $\alpha(t) = (c_1^h \circ \varphi)(t)$ satisfies $\alpha(t_0) = (c_1^h \circ \varphi)(t_0) = c_1^h(s_0) = q$, $\pi(\alpha(t)) = (c_1 \circ \varphi)(t) = c_2(t)$ and

$$\dot{\alpha}(t) = \dot{\varphi}(t)\dot{c}_1^h(\varphi(t)) \in \mathcal{H}_{\alpha(t)},$$

since $\dot{c}_1^h(\varphi(t)) \in \mathcal{H}_{\alpha(t)}$. Therefore, $\alpha(t)$ is the horizontal lift c_2^h of $c_2 : J \to \bar{Q}$ that passes through the point q. □

2.3 | Riemannian Submersions

Let (Q, g) and $(\bar{Q}, \bar{g})$ be Riemannian manifolds and the map $\pi : Q \to \bar{Q}$ a surjective submersion. Let $V\pi := \ker T\pi$ and $\mathcal{D} := V^\perp\pi$ denote the vertical and horizontal distributions, respectively. Then, the map π is said to be a Riemannian submersion if

$$\bar{g}(T\pi(X), T\pi(Y)) = g(X, Y)$$

for $X, Y \in \mathcal{D}$ (see [23–25] for more details). Since π is a submersion, $T_q\pi : (\mathcal{D}_q, g(q)|_{\mathcal{D}\times\mathcal{D}}) \to (T_{\pi(q)}\bar{Q}, \bar{g}(\pi(q)))$ is a linear isometry for all $q \in Q$. So, for each $X \in \mathfrak{X}(\bar{Q})$ there is a unique horizontal vector field $X^h \in \Gamma(\mathcal{D})$ such that $T\pi(X^h) = X$. The vector field X^h is the horizontal lift of X by the Riemannian submersion π.

Lemma 2.4. *If $X, Y \in \mathfrak{X}(\bar{Q})$, then*

1. $g(X^h, Y^h) = \bar{g}(X, Y) \circ \pi$.
2. $\mathcal{P}([X^h, Y^h]) = ([X, Y])^h$, *where $\mathcal{P} : TQ \to \mathcal{D}$ is the orthogonal projection.*
3. $\mathcal{P}(\nabla^g_{X^h} Y^h) = (\nabla^{\bar{g}}_X Y)^h$, *where ∇^g and $\nabla^{\bar{g}}$ are the Levi-Civita connections of (Q, g) and $(\bar{Q}, \bar{g})$, respectively.*

Proof. See Lemma 45, Ch. 7 in [25]. □

A simple corollary of the previous lemma is that geodesics $c : I \to Q$ satisfying $\dot{c}(t) \in \mathcal{D}_{c(t)}$ project under π to geodesics in $\bar{Q}$.

Moreover, the tangent lift of geodesics in Q with initial velocity in $\mathcal{D}$ always remains in $\mathcal{D}$. We include a proof of this fact for completeness. We will use the orthogonal projections $\mathcal{P} : TQ \to \mathcal{D}$ and $\mathcal{Q} : TQ \to V\pi$ in the proof of the following result.

Proposition 2.5. *Let $\pi : Q \to \bar{Q}$ be a Riemannian submersion with $\mathcal{D} = V^\perp\pi$ and $V\pi = \ker T\pi$ the horizontal and vertical distributions. If c is a geodesic of Q which is horizontal at one point, then it is always horizontal. In fact, the horizontal lift of a geodesic in $(\bar{Q}, \bar{g})$ is a geodesic for the Riemannian manifold (Q, g).*

Proof. Let $c : [0, h] \to Q$ be a geodesic of Q satisfying $c(0) = q_0$, $\dot{c}(0) \in \mathcal{D}_{q_0}$. Consider the projection $\bar{c} : [0, h] \to \bar{Q}$ defined by $\bar{c} = \pi \circ c$. In the following note that $\dot{c}$ might be written as

$$\dot{c} = \mathcal{P}(\dot{c}) + \mathcal{Q}(\dot{c}),$$

that is, the sum of the projection to the horizontal and vertical spaces. The length of the curve c is the functional

$$\ell(c) = \int_0^h \|\dot{c}\|\, dt,$$

and if h is sufficiently small it is the Riemannian distance between $q_0 := c(0)$ and $q_1 := c(h)$. Attending to the decomposition of $\dot{c}$ we have that

$$\ell(c) \geqslant \int_0^h \|\mathcal{P}(\dot{c})\|\, dt = \int_0^h \|T\pi \circ \mathcal{P}(\dot{c})\|\, dt,$$

where the second equality comes from the fact that the metric preserves the inner product of horizontal tangent vectors. Since vertical vectors are by definition in the kernel of $T\pi$, we have that $T\pi(\mathcal{P}(\dot{c})) = T\pi(\dot{c})$. Therefore, we conclude that

$$\ell(c) \geqslant \int_0^h \|(\pi \circ c)'\|\, dt = \ell(\bar{c}).$$

Denote by $\bar{c}^h : [0, h] \to Q$ the horizontal lift of the curve $\bar{c}$ to the point q_0. Since $\bar{c}^h$ is a horizontal curve, we can easily deduce that $\ell(\bar{c}^h) = \ell(\bar{c})$. Therefore, $\bar{c}^h$ is a curve on Q joining q_0 and q_1 and satisfying

$$\ell(c) \geqslant \ell(\bar{c}^h).$$

Since c is a geodesic we conclude that at least for sufficiently small h, $c = \bar{c}^h$. □

Remark 2.6. The geodesic relation in Proposition 2.5 is also a direct consequence of the following equation:

$$\nabla^g_{X^h} Y^h = \left(\nabla^{\bar{g}}_X Y\right)^h + \frac{1}{2} Q[X^h, Y^h]$$

(see Lemmas 2 and 3 in [23]).

2.4 | Generalized Chaplygin System

A *generalized Chaplygin system* ([9–11, 33]) is a nonholonomic system whose configuration manifold Q is a principal G-bundle $\pi : Q \to Q/G$, the constraint distribution $\mathcal{D}$ determines a principal connection on the bundle with connection form ω and the Lagrangian function $L : TQ \to \mathbb{R}$ is a G-invariant regular function for the *lifted action* of G on TQ, that is,

$$L((T_q\Phi_g)(v)) = L(v), \quad \forall\, v \in T_qQ.$$

The *lifted action* is the map $\Phi^{TQ} : G \times TQ \to TQ$, defined by $\Phi^{TQ}(g, v) = (T_q\Phi_g)(v)$. If the action is free and proper then the lifted action also is so.

On the space $\bar{Q} = Q/G$, one defines a connection (see [9]):

$$\bar{\nabla}_{\bar{X}} \bar{Y} = T\pi \left(\nabla^{nh}_{\bar{X}^h} \bar{Y}^h\right) \quad \text{for } \bar{X}, \bar{Y} \in \mathfrak{X}(\bar{Q}). \tag{4}$$

Here, $\bar{X}^h$, $\bar{Y}^h$ are the horizontal lifts of the vector fields $\bar{X}$, $\bar{Y}$, respectively, with respect to the principal connection ω, that is, $\omega(\bar{X}^h) = \omega(\bar{Y}^h) = 0$ and $\bar{X}^h$, $\bar{Y}^h$ are π-projectable over $\bar{X}$, $\bar{Y}$, respectively.

Proposition 2.7. *The geodesics of ∇^{nh} starting in $\mathcal{D}$ project onto geodesics of $\bar{\nabla}$. Conversely, let $\bar{c}$ be a geodesic of $\bar{\nabla}$ and choose $q \in Q$ such that $\pi(q) = \bar{c}(0)$. Then, the geodesic of ∇^{nh} starting in q is the horizontal lift of $\bar{c}$ with respect to the principal connection ω.*

The proof of Proposition 2.7 is a direct consequence that $\mathcal{D}$ is geodesically invariant with respect to the nonholonomic affine connection ∇^{nh}.

3 | Kinetic Hamiltonization of ϕ-Simple Kinetic Chaplygin Systems: The Modified Metric

Throughout this section, suppose that (Q, g) is a Riemannian manifold, $\mathcal{D}$ is a distribution on Q, and consider the kinetic nonholonomic system defined by the Lagrangian function L_g and also by the distribution $\mathcal{D}$.

In addition, suppose that the nonholonomic system $(L_g, \mathcal{D})$ is a generalized Chaplygin system with respect to the free and proper action of a Lie group G given by the map $\Phi : G \times Q \to Q$. This means that the Riemannian metric g is G-invariant and $\mathcal{D}$ is the horizontal subbundle associated with a principal connection ω.

Denote by $\bar{Q}$ the reduced space Q/G and by $\pi : Q \to \bar{Q}$ the projection. Under this hypothesis, and following the discussion in Section 2.4, the nonholonomic vector field reduces to a vector field on $T\bar{Q}$. We will consider on $\bar{Q}$ the Riemannian metric $\bar{g}$ defined by

$$\bar{g}(\bar{X}, \bar{Y}) \circ \pi = g(\bar{X}^h, \bar{Y}^h), \ \text{for } \bar{X}, \bar{Y} \in \mathfrak{X}(\bar{Q}).$$

Define the gyroscopic tensor $\mathcal{T}$ as a (1,2)-tensor field $\mathcal{T} : \mathfrak{X}(\bar{Q}) \times \mathfrak{X}(\bar{Q}) \to \mathfrak{X}(\bar{Q})$ given by

$$\mathcal{T}(\bar{X}, \bar{Y})(\pi(q)) = T_q\pi\left(\mathcal{P}[\bar{X}^h, \bar{Y}^h](q)\right) - [\bar{X}, \bar{Y}](\pi(q)).$$

On the reduced space $\bar{Q}$, take coordinates $(\bar{q}^a)$.

The gyroscopic tensor is given in the previous coordinates by

$$\mathcal{T}\left(\frac{\partial}{\partial \bar{q}^a}, \frac{\partial}{\partial \bar{q}^b}\right) = C^c_{ab}(\bar{q}) \frac{\partial}{\partial \bar{q}^c},$$

where C^c_{ab} is given by

$$\mathcal{P}\left(\left[\left(\frac{\partial}{\partial \bar{q}^a}\right)^h, \left(\frac{\partial}{\partial \bar{q}^b}\right)^h\right]\right) = C^c_{ab}(\bar{q}) \left(\frac{\partial}{\partial \bar{q}^c}\right)^h,$$

with $\mathcal{P} : TQ \to \mathcal{D}$ the orthogonal projection (see [6] and the references therein). Suppose that $\Gamma_{(g,\mathcal{D})} \in \mathfrak{X}(\mathcal{D})$ is the vector field on $\mathcal{D}$ whose trajectories are the nonholonomic trajectories of the Chaplygin system associated with the Lagrangian function L_g. Denote by Ω_{nh} the almost symplectic 2-form on $T^*\bar{Q}$ given by

$$\Omega_{nh} = \omega_{\bar{Q}} + \Omega_{\mathcal{T}},$$

where

$$\Omega_{\mathcal{T}}(\alpha)(U, V) = \alpha\left(\mathcal{T}\left(T_\alpha\pi_{\bar{Q}}(U), T_\alpha\pi_{\bar{Q}}(V)\right)\right), \ \text{for } \alpha \in T^*\bar{Q},\ U, V \in T_\alpha(T^*\bar{Q}),$$

$\pi_{\bar{Q}} : T^*\bar{Q} \to \bar{Q}$ is the canonical projection and $\omega_{\bar{Q}}$ denotes the canonical symplectic form on $T^*\bar{Q}$. Denote also by $X_{nh} \in \mathfrak{X}(T^*\bar{Q})$ the vector field which is obtained as the projection of $\Gamma_{(g,\mathcal{D})}$ by the smooth map

$$\flat_{\bar{g}} \circ T\pi : \mathcal{D} \to T\bar{Q} \to T^*\bar{Q},$$

where $\flat_{\bar{g}} : T\bar{Q} \to T^*\bar{Q}$ is the musical isomorphism associated with the metric $\bar{g}$. X_{nh} is characterized by the following condition:

$$i_{X_{nh}} \Omega_{nh} = dH_{\bar{g}},$$

with $H_{\bar{g}} : T^*\bar{Q} \to \mathbb{R}$ being the kinetic energy induced by the Riemannian metric $\bar{g}$, that is,

$$H_{\bar{g}}(\alpha) = \frac{1}{2} \bar{g}(\sharp_{\bar{g}}(\alpha), \sharp_{\bar{g}}(\alpha)), \tag{5}$$

where $\sharp_{\bar{g}} : T^*\bar{Q} \to T\bar{Q}$ is the inverse musical isomorphism $\sharp_{\bar{g}} = \left(\flat_{\bar{g}}\right)^{-1}$. This almost symplectic 2-form can be used to define in a natural way an associated linear almost Poisson structure (for more details see [6]).

Definition 3.1 [6]. A ϕ-simple system, with $\phi \in C^\infty(\bar{Q})$, is a nonholonomic Chaplygin system for which the gyroscopic tensor satisfies

$$\mathcal{T}(Y,Z) = Z(\phi)Y - Y(\phi)Z, \ \forall\, Y,Z \in \mathfrak{X}(\bar{Q}).$$

ϕ-Simple systems have the remarkable property that the 2-form

$$\Omega = \exp(\phi \circ \pi_{\bar{Q}})\Omega_{nh}$$

is a symplectic structure (see [6]).

Next, we present two known results in the literature (see, e.g., Theorem 3.5 in [2]) for which we provide an intrinsic proof in order to make the paper as self-contained as possible. They state that an appropriate rescaling of time and momenta puts a Hamiltonizable nonholonomic Chaplygin system in standard Hamiltonian form.

Proposition 3.2. *If a Chaplygin system is ϕ-simple then the vector bundle isomorphism $\psi : T^*\bar{Q} \to T^*\bar{Q}$ over the identity of $\bar{Q}$ given by*

$$\psi(\alpha_{\bar{q}}) = \exp(\phi(\pi_{\bar{Q}}(\alpha_{\bar{q}})))\alpha_{\bar{q}}, \ \textit{for}\ \alpha_{\bar{q}} \in T^*_{\bar{q}}\bar{Q}$$

satisfies

$$\psi^*\omega_{\bar{Q}} = \Omega$$

and

$$H_{\bar{h}} = H_{\bar{g}} \circ \psi^{-1}, \tag{6}$$

where $\bar{h}$ is the Riemannian metric defined by

$$\bar{h} = \exp(2\phi)\bar{g}, \tag{7}$$

and $H_{\bar{h}}$ is the kinetic energy on $T^*\bar{Q}$ associated with $\bar{h}$.

Proof. From Theorem 3.21 and Lemma 3.24 in [6], a system is ϕ-simple if and only if

$$\Omega_{\mathcal{T}} = \theta_{\bar{Q}} \wedge d(\phi \circ \pi_{\bar{Q}}),$$

where $\theta_{\bar{Q}}$ is the Liouville 1-form on T^*Q. This implies that

$$\Omega = \exp(\phi \circ \pi_{\bar{Q}})\omega_{\bar{Q}} + \exp(\phi \circ \pi_{\bar{Q}})\theta_{\bar{Q}} \wedge d(\phi \circ \pi_{\bar{Q}}).$$

On the other hand, it is easy to show that

$$\psi^*\theta_{\bar{Q}} = \exp(\phi \circ \pi_{\bar{Q}})\theta_{\bar{Q}}.$$

From here, it follows that

$$\psi^*\omega_{\bar{Q}} = -d(\psi^*\theta_{\bar{Q}}) = \exp(\phi \circ \pi_{\bar{Q}})\omega_{\bar{Q}} + \exp(\phi \circ \pi_{\bar{Q}})\theta_{\bar{Q}} \wedge d(\phi \circ \pi_{\bar{Q}}) = \Omega.$$

In addition, a direct computation using the definitions of $H_{\bar{h}}$ and $H_{\bar{g}}$ proves the equality in (6). □

Now, we can prove the following result.

Corollary 3.3. *If $X_{\bar{h}} \in \mathfrak{X}(T^*\bar{Q})$ is the geodesic vector field of $\bar{h}$ in $T^*\bar{Q}$ then the vector field $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ is the push-forward of $X_{\bar{h}}$ by ψ. So, the trajectories of X_{nh} are reparameterizations of geodesics with respect to the metric $\bar{h}$.*

Proof. From Theorems 3.11 and 3.21 in [6], we have that the dynamical system X_{nh} is Hamiltonizable up to reparameterization. In fact, it satisfies

$$i_{\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}}\Omega = dH_{\bar{g}}.$$

Using the previous proposition, we conclude that

$$i_{\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}}\psi^*\omega_{\bar{Q}} = dH_{\bar{g}},$$

which leads to

$$i_{(\exp(-(\phi \circ \pi_{\bar{Q}}))T\psi \circ X_{nh} \circ \psi^{-1})}\omega_{\bar{Q}} = d(H_{\bar{g}} \circ \psi^{-1}).$$

Thus, using again Proposition 3.2, it follows that the vector field $\exp(-(\phi \circ \pi_{\bar{Q}}))T\psi \circ X_{nh} \circ \psi^{-1} \in \mathfrak{X}(T^*\bar{Q})$ is just $X_{\bar{h}}$.

This implies that

$$X_{\bar{h}} \circ \psi = T\psi \circ \big(\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}\big)$$

and the vector field $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ is the push-forward of $X_{\bar{h}}$ by ψ. So, if $\bar{\gamma} : I \to T^*\bar{Q}$ is an integral curve of $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ then $\psi \circ \bar{\gamma} : I \to T^*\bar{Q}$ is an integral curve of $X_{\bar{h}}$. Thus, using that the trajectories of $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ and $X_{\bar{h}}$ are the $\pi_{\bar{Q}}$-projections of their integral curves and the fact that $\pi_{\bar{Q}} \circ \psi = \pi_{\bar{Q}}$, we conclude that the trajectories of $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ are just the geodesics of the metric $\bar{h}$. Finally, since the trajectories of $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ are reparameterizations of the trajectories of X_{nh}, we deduce the result. □

Next, we will prove that unreduced ϕ-simple kinetic Chaplygin systems admit kinetic Hamiltonization.

Theorem 3.4. *Let $(Q, g, G, \mathcal{D})$ be a kinetic Chaplygin system and $\pi : Q \to \bar{Q}$ the projection onto the quotient space. If the system is ϕ-simple, there exists a Riemannian metric h on Q such that all its geodesics with initial velocity in $\mathcal{D}$ are reparameterizations of the nonholonomic trajectories associated with $(L_g, \mathcal{D})$. In fact, h may be chosen as follows:*

1. *$h(X,Y) = \exp(2\phi \circ \pi)g(X,Y), \forall X, Y \in \Gamma(\mathcal{D})$;*
2. *$h(Z,W) = g(Z,W), \forall Z, W \in \Gamma(V\pi)$;*
3. *$h(X,Z) = 0, \forall X \in \Gamma(\mathcal{D}),\ Z \in \Gamma(V\pi)$.*

Here, $V\pi$ is the vertical bundle associated to π.

Proof. The proof has two parts. First, we will prove that the projection $\pi : (Q,h) \to (\bar{Q}, \bar{h})$ is a Riemannian submersion, where $\bar{h}$ is the Riemannian metric defined in Proposition 3.2.

The metric h is indeed a Riemannian metric, it is clearly a symmetric (0,2)-type smooth tensor. It is positive definite since if $q \in Q$ and $Z \in T_qQ$, $Z_q \neq 0$, then

$$h(Z_q, Z_q) = h(P_{\mathcal{D}}(Z_q) + P_{V\pi}(Z_q), P_{\mathcal{D}}(Z_q) + P_{V\pi}(Z_q)),$$

where $P_{\mathcal{D}} : TQ \to \mathcal{D}$ and $P_{V\pi} : TQ \to V\pi$ are the projections associated with the decomposition $TQ = \mathcal{D} \oplus V\pi$. Hence,

$$h(Z_q, Z_q) = \bar{h}(T\pi(P_{\mathcal{D}}(Z_q)), T\pi(P_{\mathcal{D}}(Z_q))) + g(P_{V\pi}(Z_q), P_{V\pi}(Z_q)) > 0$$

by positive definiteness of $\bar{h}$ and g (note that if $Z_q \neq 0$ then $T\pi(P_{\mathcal{D}}(Z_q)) \neq 0$ or $P_{V\pi}(Z_q) \neq 0$). In addition, with respect to the Riemannian metric h, $\mathcal{D} = V^{\perp}\pi$. Hence, $\pi : (Q, h) \to (\bar{Q}, \bar{h})$ is a Riemannian submersion since

$$h(X, Y) = \bar{h}(T\pi(X), T\pi(Y))$$

for $X, Y \in \mathcal{D}$.

Therefore, by Proposition 2.5, the tangent lift of all geodesics of (Q, h) with initial velocity in $\mathcal{D}$ stay in $\mathcal{D}$ and such geodesics are the horizontal lift of geodesics of $(\bar{Q}, \bar{h})$.

Moreover, by Corollary 3.3, reparameterizations of geodesics with respect to the metric $\bar{h}$ are trajectories of X_{nh} and, using Proposition 2.7, the horizontal lift of the latter with respect to the nonholonomic principal connection ω are the nonholonomic trajectories. Note that the horizontal lift with respect to ω coincides with the horizontal lift with respect to the Riemannian submersion $\pi : (Q, h) \to (\bar{Q}, \bar{h})$. This fact, Proposition 2.5 and Lemma 2.3 imply that the nonholonomic trajectories must be reparameterizations of geodesics of h with initial velocity in $\mathcal{D}$. □

Remark 3.5. The construction of a Riemannian metric h in the conditions of the theorem above is by no means unique. Any metric h' rendering $\pi : (Q, h') \to (\bar{Q}, \bar{h})$ a Riemannian submersion will have the same geodesics with initial velocity in $\mathcal{D}$. In particular, the restriction of h' to vertical vector fields may be given by any bundle metric on the vertical bundle.

Definition 3.6. The Riemannian metric h in Theorem 3.4 is called the *principal Riemannian metric* with respect to (Q, g) and $(\bar{Q}, \bar{h})$.

Denote the norm with respect to the Riemannian metric h by $\|\cdot\|_h$ and by $d : Q \times Q \to \mathbb{R}$ the Riemannian distance induced by h, that is,

$$d(p, q) = \inf\{\ell(c) \mid c \in \Omega(q, p)\},$$

where $\Omega(q, p)$ is the set of all piecewise smooth curve segments in Q from q to p and, for each $c : [0, T] \to Q$ in $\Omega(q, p)$ the length $\ell(c)$ of the segment c is given by

$$\ell(c) = \int_0^T \|\dot{c}\|_h \, dt.$$

Then, the following corollary of Theorem 3.4 asserts that the length of a nonholonomic trajectory between two nearby points q and p is just the distance from q to p. In other words, nonholonomic trajectories between two nearby points minimize the Riemannian length. The proof of the corollary is an immediate consequence of Theorem 3.4 together with the facts that geodesics locally minimize the Riemannian distance and that the length of a curve is invariant under time reparameterization.

Corollary 3.7. *Let $(Q, g, G, \mathcal{D})$ be a kinetic Chaplygin system. Suppose that the system is ϕ-simple and h is the principal Riemannian metric. If $c : [0, T] \to Q$ is a nonholonomic trajectory there exists $0 < t_0 < T$ such that for all $t < t_0$ the length of the curve $c : [0, t] \to Q$ is just the Riemannian distance associated with h between $c(0)$ and $c(t)$.*

Remark 3.8. Note that in a Chaplygin system the distributions $\mathcal{D}$ and $V\pi$ are G-invariant, and, as a consequence, the principal metric h is also G-invariant. Therefore, the momentum map $J : TQ \to \mathfrak{g}^*$ defined by

$$J_\xi(v_q) = \langle J(v_q), \xi \rangle = h(v_q, \xi_Q(q)) \quad \forall\, \xi \in \mathfrak{g}, \forall v_q \in T_qQ,$$

renders J_ξ constant along the tangent lifts of the geodesics of h (which implies that J_ξ is a first integral of the nonholonomic dynamics for every $\xi \in \mathfrak{g}$). In particular, the level set $J^{-1}(0)$ is invariant. However, by construction of the metric h, we have $J^{-1}(0) = \mathcal{D}$. Thus, geodesics of h that are horizontal at one point are horizontal at all points. This fact yields an alternative proof of Proposition 2.5 for invariant metrics with orthogonal vertical and horizontal spaces.

4 | Mechanical Hamiltonization of ϕ-Simple Chaplygin Mechanical Systems: The Main Result

In our final and main result, we will see that the inclusion of an invariant potential function in the picture does not substantially change the previous results. Consider a smooth G-invariant potential function V on Q. By symmetry, this function is characterized by the existence of a smooth potential function $\bar{V}$ on $\bar{Q}$ such that $V = \bar{V} \circ \pi$.

The trajectories of the Lagrangian system associated with a Riemannian metric g and a potential function V, that is, the trajectories of the Lagrangian system whose Lagrangian function is given by

$$L_{(g,V)} = L_g - V \circ \tau_Q$$

are commonly designated by trajectories of the mechanical system associated with g and V.

We are interested in extending Theorem 3.4 to the case where there exists a potential function and thus the trajectories of the nonholonomic system $(L_{(g,V)}, \mathcal{D})$ will no longer be reparameterizations of geodesics but of trajectories of a mechanical system with respect to a different Riemannian metric. The proof makes use of the previous results and adapts them to the presence of the potential.

4.1 | Riemannian Submersions and Mechanical Lagrangian Systems With Basic Potential

First of all, one is interested in obtaining an analogous result to Proposition 2.5 for a mechanical system with Lagrangian function $L_{(g,V)}$ and for a Riemannian submersion $\pi : (Q, g) \to (\bar{Q}, \bar{g})$. On one hand, it is well-known (see, e.g. [29]) that a curve $c : I \to Q$ is a trajectory of the mechanical system determined by $L_{(g,V)}$ if and only if it satisfies

$$\nabla^g_{\dot{c}}\dot{c} = -\mathrm{grad}_g V(c(t)),$$

where $\mathrm{grad}_g V$ is the gradient vector field of V with respect to the metric g, that is,

$$g(\mathrm{grad}_g V(q), X) = dV(q)(X), \text{ for } X \in T_q Q.$$

So, using Lemma 2.4 and the G-invariance of V, it is not difficult to prove that

$$\nabla^{\bar{g}}_{\dot{\bar{c}}}\dot{\bar{c}} = -\mathrm{grad}_{\bar{g}} \bar{V}(\bar{c}(t)),$$

where $\bar{c} = \pi \circ c$ and c is a horizontal trajectory, that is, $\dot{c}(t) \in V^{\perp}_{c(t)}\pi$, $\forall t$ and $\bar{V} \in C^{\infty}(\bar{Q})$ is defined by $V = \bar{V} \circ \pi$. Therefore, horizontal trajectories of the mechanical system associated with $L_{(g,V)}$ project to trajectories of the mechanical system associated with $L_{(\bar{g},\bar{V})}$.

On the other hand, the vector field $\Gamma_{(g,V)}$ on TQ generating the dynamics associated with $L_{(g,V)}$, is tangent to the horizontal distribution $\mathcal{D} = V^{\perp}\pi$ implying that if c is a mechanical trajectory which is horizontal at one point then it is horizontal at all points. This can be seen by noting that this vector field can be written as

$$\Gamma_{(g,V)} = \Gamma_g - (\mathrm{grad}_g V)^{\mathrm{v}},$$

where Γ_g is the geodesic vector field of g and

$$(\mathrm{grad}_g V)^{\mathrm{v}}(v_q) = \frac{d}{dt}\Big|_{t=0}(v_q + t\,\mathrm{grad}_g V(q)), \text{ for } v_q \in T_q Q.$$

Now, it is easy to see that $\mathrm{grad}_g V$ is a horizontal vector field. In fact, if $X \in V_q\pi = \ker(T_q\pi)$ then, using that $V = \bar{V} \circ \pi$, we deduce that

$$g(\mathrm{grad}_g V(q), X) = X(V) = X(\bar{V} \circ \pi) = 0.$$

Thus, $(\mathrm{grad}_g V)^{\mathrm{v}}|_{\mathcal{D}} \in \mathfrak{X}(\mathcal{D})$. On the other hand, by Proposition 2.5, we have that $\Gamma_g|_{\mathcal{D}} \in \mathfrak{X}(\mathcal{D})$. Therefore, $\Gamma_{(g,V)}|_{\mathcal{D}} \in \mathfrak{X}(\mathcal{D})$.

In summary, we have proved the following result.

Proposition 4.1. *Let $L_{(g,V)} : TQ \to \mathbb{R}$ be a mechanical Lagrangian function and $\pi : (Q, g) \to (\bar{Q}, \bar{g})$ a Riemannian submersion such that $V \in C^{\infty}(Q)$ is a π-basic function, that is, there exists $\bar{V} \in C^{\infty}(\bar{Q})$ and $V = \bar{V} \circ \pi$. Denote by $V\pi = \ker T\pi$ and $\mathcal{D} = V^{\perp}\pi$ the vertical and horizontal distributions, respectively, associated with the Riemannian submersion π.*

1. *If $c : I \to Q$ is a horizontal trajectory for the mechanical system determined by g and V then $\bar{c} = \pi \circ c$ is a trajectory for the mechanical system determined by $\bar{g}$ and $\bar{V}$.*
2. *If $c : I \to Q$ is a trajectory for $L_{(g,V)}$ which is horizontal at one point, then it is always horizontal. In fact, the horizontal lift of a trajectory for $L_{(\bar{g},\bar{V})}$ is a trajectory for $L_{(g,V)}$.*

4.2 | The Main Result of the Paper

Before presenting the main result of this section and the paper, there are a few considerations about ϕ-simple mechanical systems that we must recall.

The notion of ϕ-simplicity is independent of the potential function. So, if $(Q, g, G, \mathcal{D})$ is a ϕ-simple kinetic nonholonomic Chaplygin system, then the mechanical system continues to be ϕ-simple under the addition of a G-invariant potential (see [6]).

Following [6], the framework described at the beginning of Section 3, still holds under the addition of a G-invariant potential function.

Let $(Q, g, V, G, \mathcal{D})$ be a ϕ-simple Chaplygin mechanical system. Consider the Hamiltonian function $H_{(g,V)} : T^*Q \to \mathbb{R}$ given by

$$H_{(g,V)}(\alpha_q) = \frac{1}{2}g(\sharp_g(\alpha_q), \sharp_g(\alpha_q)) + V(q), \text{ for } \alpha_q \in T^*_q Q.$$

Next, we define the reduced Hamiltonian function $H_{(\bar{g},\bar{V})} : T^*\bar{Q} \to \mathbb{R}$ as follows:

$$H_{(\bar{g},\bar{V})}(\alpha_{\bar{q}}) = \frac{1}{2}\bar{g}(\sharp_{\bar{g}}(\alpha_{\bar{q}}), \sharp_{\bar{g}}(\alpha_{\bar{q}})) + \bar{V}(\bar{q}), \text{ for } \alpha_{\bar{q}} \in T^*_{\bar{q}}\bar{Q},$$

where $\bar{g}$ is the reduced Riemannian metric and $\bar{V} \in C^{\infty}(\bar{Q})$ is such that $V = \bar{V} \circ \pi$.

Now, we can consider the symplectic vector bundle isomorphism $\psi : (T^*\bar{Q}, \omega_{\bar{Q}}) \to (T^*\bar{Q}, \Omega)$ in Proposition 3.2 and, consequently, we can define the Riemannian metric $\bar{h}$ on $\bar{Q}$ as in this proposition. Therefore, we can introduce a new Hamiltonian function $H_{(\bar{h},\bar{V})} : T^*\bar{Q} \to \mathbb{R}$ given by

$$H_{(\bar{h},\bar{V})}(\alpha_{\bar{q}}) = \frac{1}{2}\bar{h}(\sharp_{\bar{h}}(\alpha_{\bar{q}}), \sharp_{\bar{h}}(\alpha_{\bar{q}})) + \bar{V}(\bar{q}),$$

which is the mechanical energy associated with the Riemannian metric determined by $\bar{h}$ and the potential energy $\bar{V} \in C^{\infty}(\bar{Q})$.

Analogously to the case without potential, this Hamiltonian function satisfies

$$H_{(\bar{h},\bar{V})} = H_{(\bar{g},\bar{V})} \circ \psi^{-1}.$$

Since the system is ϕ-simple, the reduced nonholonomic vector field X_{nh} is still Hamiltonizable with the addition of the G-invariant potential function (see Theorems 3.11 and 3.21 in [6]). Therefore, the proof of Corollary 3.3 holds by replacing the Hamiltonian function $H_{\bar{h}}$ with $H_{(\bar{h},\bar{V})}$.

So, we deduce the following result.

Corollary 4.2. *If $X_{(\bar{h},\bar{V})} \in \mathfrak{X}(T^*\bar{Q})$ is the dynamical vector field associated with the mechanical Hamiltonian function $H_{(\bar{h},\bar{V})}$, then the vector field $\exp(-(\phi \circ \pi_{\bar{Q}}))X_{nh}$ is the push-forward of $X_{(\bar{h},\bar{V})}$ by ψ. So, the trajectories of the reduced nonholonomic vector field X_{nh} are reparameterizations of trajectories of the Hamiltonian system with mechanical Hamiltonian function $H_{(\bar{h},\bar{V})}$.*

Now, using Proposition 4.1, Corollary 4.2, and proceeding as in the proof of Theorem 3.4, we can prove a version of this last theorem for the more general case of a ϕ-simple Chaplygin mechanical system.

Theorem 4.3. *Let $(Q, g, V, G, \mathcal{D})$ be a Chaplygin system with G-invariant potential V. If the system is ϕ-simple, there exists a Riemannian metric h on Q such that all the mechanical trajectories associated with h and V with initial velocity in $\mathcal{D}$ are reparameterizations of the nonholonomic trajectories associated with $(L_{(g,V)}, \mathcal{D})$. In fact, h may be chosen as the principal Riemannian metric.*

Remark 4.4. Note that a more general version of Theorem 4.3 may be proved if we replace, in the statement of the theorem, the hypothesis that the Chaplygin system is ϕ-simple with the following weaker condition: there exists a smooth function $\phi \in C^\infty(\bar{Q})$ such that the trajectories of the reduced nonholonomic vector field X_{nh} are reparameterizations of the trajectories of the mechanical system with Lagrangian function $L_{(\bar{h},\bar{V})}$, where $\bar{h} = \exp(2\phi)\bar{g}$. In particular, this condition holds if X_{nh} is just the dynamics in the Hamiltonian side associated with the mechanical system whose Lagrangian function is $L_{(\bar{g},\bar{V})}$.

Remark 4.5. We have presented a proof of the main results, Theorem 3.4 (in Section 3) and Theorem 4.3, in the Hamiltonian side of the theory. But, using Corollary 4.2, these results may also be proved in the Lagrangian side. In fact, if $(Q, g, V, G, \mathcal{D})$ is a ϕ-simple Chaplygin system with G-invariant potential V, h is the principal Riemannian metric constructed in Theorem 3.4, $\tau_Q : TQ \to Q$ is the canonical tangent projection on Q and $\pi : Q \to \bar{Q}$ is the principal bundle projection, then the tangent map of the transformation $\tilde{\psi} : \mathcal{D} \to \mathcal{D}$ sending $v_q \mapsto \exp(-\phi(\pi(q)))v_q$ will map the vector field $\exp(-(\phi\circ\pi\circ(\tau_Q)|_{\mathcal{D}}))\Gamma_{(g,V,\mathcal{D})}$, where $\Gamma_{(g,V,\mathcal{D})}$ is the vector field determining the nonholonomic trajectories of this system, into the mechanical vector field $\Gamma_{(h,V)}$ restricted to the distribution $\mathcal{D}$.

We will give a sketch of the proof of this result. Denote by $(T\pi)|_{\mathcal{D}} : \mathcal{D} \to T\bar{Q}$ the restriction to $\mathcal{D}$ of the tangent map of $\pi : Q \to \bar{Q}$ and by $\Gamma_{nh} \in \mathfrak{X}(T\bar{Q})$ the $(T\pi)|_{\mathcal{D}}$-projection of the vector field $\Gamma_{(g,V,\mathcal{D})}$ (note that $\Gamma_{(g,V,\mathcal{D})}$ is G-invariant and, thus, it is $(T\pi)|_{\mathcal{D}}$-projectable). If $\bar{h}$ is the Riemannian metric on $\bar{Q}$ given by $\bar{h} = \exp(2\phi)\bar{g}$, $\bar{V} \in C^\infty(\bar{Q})$ is such that $V = \bar{V}\circ\pi$ and $\Gamma_{(\bar{h},\bar{V})}$ is the dynamical vector field in $T\bar{Q}$ associated with the mechanical Lagrangian system $L_{(\bar{h},\bar{V})}$ then, using Corollary 4.2, we deduce that these vector fields are related by the vector bundle isomorphism

$$\tilde{\bar{\psi}} = \sharp_{\bar{h}}\circ\psi\circ\flat_{\bar{g}} : T\bar{Q} \to T^*\bar{Q} \to T^*\bar{Q} \to T\bar{Q}.$$

Here, $\flat_{\bar{g}} : T\bar{Q} \to T^*\bar{Q}$, $\sharp_{\bar{h}} : T^*\bar{Q} \to T\bar{Q}$ are the musical isomorphisms induced by the Riemannian metrics $\bar{g}$ and $\bar{h}$, respectively, and $\psi : T^*\bar{Q} \to T^*\bar{Q}$ is the isomorphism in Proposition 3.2. Now, a direct computation proves that

$$\tilde{\bar{\psi}}(v_{\bar{q}}) = \exp(-\phi(\bar{q}))v_{\bar{q}}, \text{ for } v_{\bar{q}} \in T_{\bar{q}}\bar{Q}.$$

On the other hand, considering $(\tau_Q)|_{\mathcal{D}}$ the restriction of τ_Q to $\mathcal{D}$, the trajectories in Q of the vector fields $\exp(-(\phi\circ\pi\circ(\tau_Q)|_{\mathcal{D}}))\Gamma_{(g,V,\mathcal{D})} \in \mathfrak{X}(\mathcal{D})$ and $\Gamma_{(h,V)}|_{\mathcal{D}} \in \mathfrak{X}(\mathcal{D})$ are the horizontal lifts, with respect to the Riemannian submersion $\pi : (Q, h) \to (\bar{Q}, \bar{h})$, of the trajectories in $\bar{Q}$ of the vector fields $\exp(-(\phi\circ\tau_{\bar{Q}}))\Gamma_{nh} \in \mathfrak{X}(T\bar{Q})$ and $\Gamma_{(\bar{h},\bar{V})} \in \mathfrak{X}(T\bar{Q})$, respectively, where $\tau_{\bar{Q}} : T\bar{Q} \to \bar{Q}$ is the canonical tangent projection on $\bar{Q}$. Therefore, using that the following diagram is commutative,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\;\tilde{\psi}\;} & \mathcal{D} \\ {\scriptstyle (T\pi)|_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle (T\pi)|_{\mathcal{D}}} \\ T\bar{Q} & \xrightarrow{\;\tilde{\bar{\psi}}\;} & T\bar{Q} \end{array}$$

we conclude that the vector field $\exp(-(\phi\circ\pi\circ(\tau_Q)|_{\mathcal{D}}))\Gamma_{(g,V,\mathcal{D})} \in \mathfrak{X}(\mathcal{D})$ is the push-forward of $\Gamma_{(h,V)} \in \mathfrak{X}(\mathcal{D})$ by $\tilde{\psi}$.

5 | Examples

Example 1 (Vertical rolling disk). The vertical rolling disk or the rolling penny is one of the most elementary examples of a nonholonomic kinetic system. In the configuration manifold $Q = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, we introduce the local coordinate system (x, y, θ, φ) and the kinetic Lagrangian function $L_g : TQ \to \mathbb{R}$ given by

$$L_g(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

with

$$g = mdx^2 + mdy^2 + Id\theta^2 + Jd\varphi^2$$

and subjected to the distribution $\mathcal{D} \subseteq TQ$ determined by the local expression

$$\dot{x} = R\dot{\theta}\cos\varphi, \quad \dot{y} = R\dot{\theta}\sin\varphi,$$

where R is the radius of the disk, m is its mass, and I, J are the inertia.

It is well-known that under the translation

$$\Phi : \mathbb{R}^2 \times Q \longrightarrow Q$$
$$((a, b), (x, y, \theta, \varphi)) \mapsto (x + a, y + b, \theta, \varphi)$$

the kinetic nonholonomic system $(L_g, \mathcal{D})$ is a Chaplygin system with principal bundle projection $\pi : Q \to \bar{Q} := \mathbb{S}^1 \times \mathbb{S}^1$ and the gyroscopic tensor $\mathcal{T}$ vanishes identically (see [6]). So, the reduced equations are geodesics with respect to the reduced metric $\bar{g}$:

$$\bar{g} = (I + mR^2)d\theta^2 + Jd\varphi^2$$

on $\bar{Q}$. Therefore, we are in the conditions described in Remark 4.4 and we may define the principal Riemannian metric h on Q, that is, the one such that $h|_{\mathcal{D}\times\mathcal{D}} = g|_{\mathcal{D}\times\mathcal{D}}$, $h|_{V\pi\times V\pi} = g|_{V\pi\times V\pi}$ and $h|_{\mathcal{D}\times V\pi} = 0$. Let $\{e_1, e_2, e_3, e_4\}$ be the basis of vector fields on Q defined by

$$e_1 = \frac{\partial}{\partial\theta} + R\cos\varphi\frac{\partial}{\partial x} + R\sin\varphi\frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial\varphi},\ e_3 = \frac{\partial}{\partial x},\ e_4 = \frac{\partial}{\partial y}$$

so that $\mathcal{D} = \langle\{e_1, e_2\}\rangle$ and $V\pi = \langle\{e_3, e_4\}\rangle$. Therefore, in $V\pi \times V\pi$ we have

$$h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = m,\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = m,\ h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0. \quad (8)$$

In $\mathcal{D} \times \mathcal{D}$, we must have

$$h(e_1, e_1) = g(e_1, e_1) = I + mR^2,\ h(e_2, e_2) = g(e_2, e_2) = J,$$
$$h(e_1, e_2) = g(e_1, e_2) = 0.$$

So, using (8) we may write the following three equations:

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) + 2R\cos\varphi\, h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial\theta}\right) + 2R\sin\varphi\, h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\right) = I$$
$$h\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right) = J$$
$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right) + R\cos\varphi\, h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial\varphi}\right) + R\sin\varphi\, h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial\varphi}\right) = 0.$$

Finally, from the condition that $\mathcal{D}$ and $V\pi$ must be orthogonal under h, we deduce

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}\right) = -mR\cos\varphi, \quad h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y}\right) = -mR\sin\varphi$$
$$h\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial x}\right) = 0, \quad h\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial y}\right) = 0.$$

Inserting the last four equations into the previous three, we deduce

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = I + 2mR^2, \quad h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right) = 0.$$

Therefore, the modified metric h is

$$h = mdx^2 + mdy^2 + (I + 2mR^2)d\theta^2 + Jd\varphi^2$$
$$- mR\cos\varphi\, dxd\theta - mR\sin\varphi\, dyd\theta.$$

The geodesic equations for the Riemannian metric h are

$$\ddot{x} = -\frac{mR^2}{2(I + mR^2)}\sin(2\varphi)\dot{x}\dot{\varphi} + \frac{mR^2}{I + mR^2}\cos^2(\varphi)\dot{y}\dot{\varphi} - R\sin(\varphi)\dot{\theta}\dot{\varphi}$$
$$\ddot{y} = -\frac{mR^2}{I + mR^2}\sin^2(\varphi)\dot{x}\dot{\varphi} + \frac{mR^2}{2(I + mR^2)}\sin(2\varphi)\dot{y}\dot{\varphi} + R\cos(\varphi)\dot{\theta}\dot{\varphi}$$
$$\ddot{\theta} = -\frac{mR}{I + mR^2}\sin(\varphi)\dot{x}\dot{\varphi} + \frac{mR}{I + mR^2}\cos(\varphi)\dot{y}\dot{\varphi}$$
$$\ddot{\varphi} = \frac{mR}{J}\sin(\varphi)\dot{x}\dot{\theta} - \frac{mR}{J}\cos(\varphi)\dot{y}\dot{\theta}$$

and their solutions with initial velocity in $\mathcal{D}$ satisfy

$$\ddot{x} = -R\sin(\varphi)\dot{\theta}\dot{\varphi}$$
$$\ddot{y} = R\cos(\varphi)\dot{\theta}\dot{\varphi}$$
$$\ddot{\theta} = 0$$
$$\ddot{\varphi} = 0.$$

On the other hand, the equations of the nonholonomic trajectories can be found using Equation (3), from where we obtain

$$m\ddot{x} = \lambda_1$$
$$m\ddot{y} = \lambda_2$$
$$I\ddot{\theta} = -\lambda_1 R\cos\varphi - \lambda_2 R\sin\varphi$$
$$J\ddot{\varphi} = 0$$

which after determination of the Lagrange multipliers give precisely the previous set of equations.

Example 2 (The nonholonomic particle). Consider a nonholonomic system with configuration space $Q = \mathbb{R}^3$, Lagrangian function given by

$$L_g = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and subjected to the constraint $\dot{z} = y\dot{x}$. This system, known as the nonholonomic particle, is a Chaplygin system together with the group action of $\mathbb{R}$ acting by translations on z. The reduced configuration space is $\bar{Q} = \mathbb{R}^2$ with principal bundle projection $\pi : Q \to \bar{Q}$ given by $\pi(x, y, z) = (x, y)$.

Moreover, from [6], it is also a ϕ-simple system with

$$\phi(x, y) = -\frac{1}{2}\ln(1 + y^2).$$

This system has the corresponding reduced Riemannian metric $\bar{g}$ on $\bar{Q}$ given by

$$\bar{g} = (1 + y^2)dx^2 + dy^2.$$

Therefore, the conformal change $\bar{h} = e^{2\phi}\bar{g}$ gives the Riemannian metric

$$\bar{h} = dx^2 + \frac{dy^2}{1 + y^2}.$$

The principal Riemannian metric on Q figuring in Theorem 3.4 is

$$h = (1 + y^2)dx^2 + \frac{dy^2}{1 + y^2} + dz^2 - ydxdz.$$

The geodesic equations with respect to h are

$$\ddot{x} = -y\dot{x}\dot{y} + \dot{y}\dot{z}$$
$$\ddot{y} = \frac{(y^4 + 2y^2 + 1)y\dot{x}^2 - (y^4 + 2y^2 + 1)\dot{x}\dot{z} + y\dot{y}^2}{1 + y^2}$$
$$\ddot{z} = -y^2\dot{x}\dot{y} + \dot{x}\dot{y} + y\dot{y}\dot{z}.$$

Their solutions with initial velocity in $\mathcal{D}$ satisfy

$$\ddot{x} = 0 \\ \ddot{y} = \frac{y\dot{y}^2}{1+y^2} \qquad (9) \\ \ddot{z} = \dot{x}\dot{y}.$$

On the other hand, using (3), we deduce that the nonholonomic trajectories are solutions of the equations

$$\ddot{x} = -y\frac{\dot{x}\dot{y}}{1+y^2} \\ \ddot{y} = 0 \qquad (10) \\ \ddot{z} = \frac{\dot{x}\dot{y}}{1+y^2}$$

with Lagrange multiplier already determined.

Indeed, if we rewrite (10) as a first-order system of differential equations, we obtain the equations

$$\dot{x} = v_x \qquad \dot{v_x} = -y\frac{v_x v_y}{1+y^2} \\ \dot{y} = v_y \qquad \dot{v_y} = 0 \\ \dot{z} = v_z \qquad \dot{v_z} = \frac{v_x v_y}{1+y^2},$$

whose solutions are trajectories of the vector field $\Gamma_{(g,\mathcal{D})}$. Consequently, the trajectories of the vector field $\sqrt{1+y^2}\Gamma_{(g,\mathcal{D})}$ satisfy

$$\dot{x} = \sqrt{1+y^2}v_x \qquad \dot{v_x} = -y\frac{v_x v_y}{\sqrt{1+y^2}} \\ \dot{y} = \sqrt{1+y^2}v_y \qquad \dot{v_y} = 0 \\ \dot{z} = \sqrt{1+y^2}v_z \qquad \dot{v_z} = \frac{v_x v_y}{\sqrt{1+y^2}}.$$

Considering the change of variables $\bar{v}_x = \sqrt{1+y^2}v_x$, $\bar{v}_y = \sqrt{1+y^2}v_y$, and $\bar{v}_z = \sqrt{1+y^2}v_z$, differentiating $\bar{v}_x$, $\bar{v}_y$, and $\bar{v}_z$ and restricting to $\mathcal{D}$, we may obtain the equations in terms of the new variables

$$\dot{x} = \bar{v}_x \qquad \dot{\bar{v}}_x = 0 \\ \dot{y} = \bar{v}_y \qquad \dot{\bar{v}}_y = \frac{y\bar{v}_y^2}{1+y^2} \\ \dot{z} = \bar{v}_z \qquad \dot{\bar{v}}_z = \bar{v}_x\bar{v}_y.$$

This system of equations is equivalent to system (9). Hence, the geodesics of h coincide with the trajectories of $\sqrt{1+y^2}\Gamma_{(g,\mathcal{D})}$ and, thus, they are a reparameterization of nonholonomic trajectories.

If a potential function V independent of z is also present, then the nonholonomic trajectories associated with $L = L_g - V$ are reparameterizations of the mechanical trajectories associated with the Lagrangian $L_h - V$ and initial velocity on $\mathcal{D}$.

Remark 5.1. In this example, as we mentioned in Remark 4.5, we have observed that the tangent map of the transformation $\bar{\psi} : \mathcal{D} \to \mathcal{D}$ sending $v_q \mapsto \exp(-\phi(\pi(q)))v_q$ is mapping the vector field $\exp(-\phi(\pi(q)))\Gamma_{(g,\mathcal{D})}$, where $\Gamma_{(g,\mathcal{D})}$ is the vector field determining nonholonomic trajectories, into the geodesic vector field Γ_h restricted to the distribution $\mathcal{D}$.

Example 3 (The Veselova problem). Consider a nonholonomic kinetic system in the manifold of three-dimensional rotation matrices $Q = SO(3)$ modeling the rotational motion of a rigid body subject to a constraint in the axis of rotation that we will describe below.

Let us write

$$\hat{\omega} = \dot{R}R^T, \qquad \hat{\Omega} = R^T\dot{R}$$

for the right and left trivializations of the velocity vector $\dot{R}$. Then $\omega, \Omega \in \mathbb{R}^3$ give the spatial and body coordinates of the angular velocity and are related by $\omega = R\Omega$. The vectors ω and Ω are related to the matrices $\hat{\omega}$ and $\hat{\Omega}$ through the hat map $\hat{(\cdot)} : \mathbb{R}^3 \to \mathfrak{so}(3)$ defined by

$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}.$$

Denote also

$$\alpha := R^T e_1, \quad \beta := R^T e_2, \quad \gamma := R^T e_3,$$

the so-called Poisson vectors which give the body coordinates of the space frame axes. Then α, β, γ are the rows of the matrix R.

The Riemannian metric g is the left-invariant metric corresponding to the inner product $(\Omega_1, \Omega_2)_{\mathbb{I}} = \Omega_1^T \mathbb{I}\Omega_2$ on $\mathfrak{so}(3)$, where Ω_i are identified with vectors in $\mathbb{R}^3$ through the hat map and $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the inertia tensor $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$. The constraint is determined by a right-invariant distribution whose value at the identity is the subspace

$$\mathfrak{d} = \{\omega \in \mathfrak{so}(3) \mid (\omega, e_3) = 0\},$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.

Nonholonomic systems on Lie groups where the Lagrangian function is left-invariant and the constraint distribution is right-invariant are usually called LR nonholonomic systems. Despite the apparent asymmetry of the system, there is a symmetry under the left action of the Lie group

$$SO(2) = \{R \in SO(3) \mid R^{-1}e_3 = e_3\},$$

that is, the group of rotations around the axis e_3. Under this action, the system is a Chaplygin system whose reduced space is $\bar{Q} = \mathbb{S}^2$ with principal bundle projection $\pi : Q \to \bar{Q}$ given by $\pi(R) = R^T e_3$. Here,

$$\mathbb{S}^2 = \{\gamma = (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The right-invariant distribution generated by the subspace $\mathfrak{d}$ of the Lie algebra happens to be invariant under the action of $SO(2)$ and thus it is a horizontal distribution for a principal connection on π. In addition, the vertical distribution $V\pi = \ker T\pi$ is the right-invariant distribution given by

$$V_R\pi = \{\hat{\omega}R \in T_R SO(3) \mid R^T\hat{\omega} \cdot e_3 = 0\}, \quad \text{for } R \in SO(3).$$

It is better described by its value at the identity

$$V_I\pi = \{\omega \in \mathbb{R}^3 \mid \omega \times e_3 = 0\} = \operatorname{span}\{e_3\},$$

where I denotes the 3×3 identity matrix.

Note that, again following [6], this system is also a ϕ-simple system with $\phi(\gamma) = -\frac{1}{2}\ln\left((\mathbb{I}^{-1}\gamma, \gamma)\right)$ and $(\cdot, \cdot)$ being the euclidean inner product on $\mathbb{R}^3$.

To give an expression for the principal Riemannian metric h, it is useful to use right-trivialized coordinates. The kinetic energy of the rigid body determined by g is left-invariant and satisfies

$$g((R, \Omega), (R, \Omega)) = (\mathbb{I}\Omega, \Omega),$$

in the left trivialization, and instead

$$g((R, \omega), (R, \omega)) = (R\mathbb{I}R^T\omega, \omega)$$

in the right trivialization. Note that

$$R\mathbb{I}R^T = \begin{pmatrix} (\mathbb{I}\alpha, \alpha) & (\mathbb{I}\alpha, \beta) & (\mathbb{I}\alpha, \gamma) \\ (\mathbb{I}\alpha, \beta) & (\mathbb{I}\beta, \beta) & (\mathbb{I}\beta, \gamma) \\ (\mathbb{I}\alpha, \gamma) & (\mathbb{I}\beta, \gamma) & (\mathbb{I}\gamma, \gamma) \end{pmatrix}.$$

Considering that the constraint distribution $\mathcal{D}$ and the vertical distribution $V\pi$ are right-invariant, it is more natural to express the principal metric h in the right trivialization. Given that $\mathfrak{d}$ corresponds to vectors having $\omega_3 = 0$ and $V_I\pi$ corresponds to vectors having $\omega_1 = \omega_2 = 0$, it follows that the principal metric h is determined in the right trivialization by the block matrix

$$\begin{pmatrix} \exp(2\phi\circ\pi)\begin{pmatrix} (\mathbb{I}\alpha, \alpha) & (\mathbb{I}\alpha, \beta) \\ (\mathbb{I}\alpha, \beta) & (\mathbb{I}\beta, \beta) \end{pmatrix} & 0 \\ 0 & (\mathbb{I}\gamma, \gamma) \end{pmatrix}.$$

Such a metric can be written as $h((R, \omega), (R, \omega)) = (H\omega, \omega)$, with

$$H = \frac{1}{(\mathbb{I}^{-1}\gamma, \gamma)}\left(I - e_3e_3^T\right)^{\top} R\mathbb{I}R^{-1}\left(I - e_3e_3^T\right) + \left(e_3e_3^T\right)^{\top} R\mathbb{I}R^{-1}\left(e_3e_3^T\right).$$

Using this expression, the Lagrangian $L_h : TQ \to \mathbb{R}$ corresponding to the principal metric h is shown to be expressed in the left trivialization as

$$L_h(R, \Omega) = \frac{1}{(\mathbb{I}^{-1}\gamma, \gamma)}\left(\frac{1}{2}(\mathbb{I}\Omega, \Omega) - (\gamma, \Omega)(\mathbb{I}\Omega, \gamma) + \frac{1}{2}(\gamma, \Omega)^2(\mathbb{I}\gamma, \gamma)\right) + \frac{1}{2}(\gamma, \Omega)^2(\mathbb{I}\gamma, \gamma).$$

The dependence of L_h on γ makes it clear that L_h is not left-invariant. On the other hand, the independence of L_h on α and β is due to the SO(2) invariance.

Now, according to Theorem 3.4, the geodesics of L_h restricted to $(\gamma, \Omega) = 0$, agree up to a time reparameterization with the solutions of the Veselova system. This can be checked using the Euler–Poincaré equations for L_h, that is,

$$\frac{d}{dt}\left(\frac{\partial L_h}{\partial \Omega}\right) = \frac{\partial L_h}{\partial \Omega} \times \Omega + \frac{\partial L_h}{\partial \gamma} \times \gamma, \qquad \dot{\gamma} = \gamma \times \Omega.$$

Remark 5.2. A multidimensional version of the Veselova problem with configuration space the Lie group $SO(n)$ was considered in [6]. This system is also ϕ-simple Chaplygin. So, their trajectories are reparameterizations of the trajectories of an unconstrained mechanical Lagrangian system on $SO(n)$.

6 | Future Work

The previous results give rise to several questions and open new research avenues. On one hand, it would be interesting to extend the results in this paper to more general Chaplygin systems that are not necessarily ϕ-simple (see Section 5 in [6]). On the other hand, if the symmetries are not just of kinematic type (as in Chaplygin systems) but they are, for example, horizontal, a natural question is if we can still find a Riemannian metric Hamiltonizing the unreduced system. In fact, there are examples of nonholonomic systems in the literature that are not Chaplygin but still their trajectories are geodesics with respect to a modified metric or even with respect to the original metric (e.g., the Chaplygin sphere [34]).

Finally, it would be interesting to take advantage of the modified Riemannian metrics to produce numerical integrators for non-holonomic systems (see [35] for a first approach and also [36, 37] in a similar direction).

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Data Availability Statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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