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Virtual constraints on Riemannian homogeneous spaces ⋆

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Abstract: Virtual constraints are relations imposed on a control system which insure invariance via feedback control, as opposed to physical constraints acting on the system. In this work, we introduce the notion of virtual constraints on Riemannian homogeneous spaces in a geometric framework which is a generalization of the classical controlled invariant distribution setting and we show the existence and uniqueness of a control law preserving the invariant distribution. We illustrate the theory with an example.

Keywords: Virtual constraints, feedback control, Riemannian homogeneous spaces, geometric control.

1. INTRODUCTION

Virtual constraints are relations on the state variables of a mechanical model that are imposed through a time-invariant feedback controller. Virtual holonomic constraints have been studied over the past few years in a variety of contexts, such as motion planning and control (see Freidovich et al. (2008), Shiriaev et al. (2010), Mohammadi et al. (2018) and Westerberg et al. (2009) for instance) and biped locomotion where it was used to achieve desired walking gaits (see Chevallereau et al. (2003) and Westervelt et al. (2018) for instance).

Control systems on Lie groups provide a general framework for a class of systems that includes controlled spacecraft and unmanned autonomous vehicles such as aerial and underwater vehicles. In general, the configuration space for these systems is globally described by a matrix Lie group. This framework gives rise to coordinate-free expressions for the dynamics describing the behavior of the system. The theory of irtual constraints on Lie groups has been recently developed in Stratoglou et al. (2023).

In this paper we discuss virtual constraints on spaces which are not necessarily Lie groups themselves, but nonetheless possess certain symmetries and invariances that allow for similar results to be obtained: *Homogeneous spaces* defined as follows. Let G be a connected Lie group. A homogeneous space H of G is a smooth manifold on which G acts transitively. Note that any Lie group is itself a homogeneous space, where the transitive action is given by left-translation (or right-translation). Any homogenous space H of G is diffeomorphic to a quotient space of the form G/K, where K is a Lie subgroup of G. This identification allowd us to translate much of the geometry of the Lie Group G into the homogeneous space H. Just as an example, Riemannian homogeneous spaces are the ones in which the projection $G \to H$ is a Riemannian submersion and much of the Riemannian geometry on G might be projected to H. In particular, nonlinear dynamics on H might be lifted and pushed forward to a dynamics in the Lie algebra of G.

In particular, a virtual constraint on a Riemannian homogeneous space H is described by a G-invariant distribution on the configuration manifold of the system for which there is a feedback control making it invariant under the flow of the closed-loop system. We provide sufficient conditions for the existence and uniqueness of such a feedback law defining the virtual constraint on a Riemannian homogeneous space. Although the result was already known from previous papers (see, e.g. Simoes et al. (2023)), we are able to explicitly construct the control law by making use of the lifted dynamics to the Lie algebra. This procedure is essential in many situations where finding out an explicit formula for the controller would be challenging. An example is the sphere, where to give an explicit formula for the control law using methods from previous papers, it would

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require working with a coordinate chart on the sphere and making sure that any singularity would be solved by a proper change of coordinates. The use of the homogeneous space structure conjugated with a proper treatment of Lie groups given in Stratoglou et al. (2023) solves that problem.

The paper is structured as follows. Section 2 introduces the background on Riemannian manifolds and we explore some geometric properties of Riemannian geometry on Lie groups. Section 3 describes Riemannian homogeneous spaces and the corresponding geodesic equations. In Section 4 we introduce virtual constraints on Riemannian homogeneous spaces, we provide sufficient conditions for the existence and uniqueness of a feedback law defining the virtual constraints. Finally in Section 5 we illustrate the theory with an example, a damped spherical pendulum.

2. BACKGROUND ON RIEMANNIAN MANIFOLDS

Let Q be an n-dimensional manifold equipped with a Riemannian metric $\langle\cdot,\cdot\rangle$, i.e., a positive-definite symmetric covariant 2-tensor field. That is, to each point $q \in Q$ we assign a positive-definite inner product $\langle\cdot,\cdot\rangle_q : T_qQ \times T_qQ \to \mathbb{R}$, where T_qQ is the *tangent space* of Q at q and $\langle\cdot,\cdot\rangle_q$ varies smoothly with respect to q. We denote by $\tau_q : T_qQ \to Q$ the smooth projection assigning to each tangent vector v_q the point q at which the vector is tangent. The length of a tangent vector is defines as $\|v_q\| = \langle v_q, v_q\rangle^{1/2}$ with $v_q \in T_qQ$. For any $p \in Q$, the Riemannian metric induces an invertible map $\flat : T_pQ \to T_p^*Q$, called the *flat map*, defined by $\flat(X)(Y) = \langle X, Y\rangle$ for all $X, Y \in T_pQ$. The inverse map $\sharp : T_p^*Q \to T_pQ$, called the *sharp map*, is similarly defined implicitly by the relation $\langle \sharp(\alpha), Y\rangle = \alpha(Y)$ for all $\alpha \in T_p^*Q$. Let $C^\infty(Q)$ and $\Gamma(TQ)$ denote the spaces of smooth scalar fields and smooth vector fields on Q, respectively. The gradient of a function on a Riemannian manifold is given by $\text{grad} f(p) = \sharp(df(p))$ for all $p \in Q$.

Vector fields are a special case of smooth sections of vector bundles. In particular, they are smooth maps of the form $X : Q \to TQ$ such that $\tau_Q \circ X = \text{id}_Q$, the identity function on Q. An *affine connection* on Q is a map $\nabla : \Gamma(TQ) \times \Gamma(TQ) \to \Gamma(TQ)$ which is $C^\infty(Q)$-linear in the first argument, $\mathbb{R}$-linear in the second argument, and satisfies the product rule $\nabla_X(fY) = X(f)Y + f\nabla_X Y$ for all $f \in C^\infty(Q)$, $X \in \Gamma(TQ)$, $Y \in \Gamma(TQ)$. The connection plays a role similar to that of the directional derivative in classical real analysis. The operator ∇_X which assigns to every smooth section Y the vector field $\nabla_X Y$ is called the *covariant derivative* (of Y) *with respect to* X.

Let $q : I \to Q$ be a smooth curve parameterized by $t \in I \subset \mathbb{R}$, and denote the set of smooth vector fields along q by $\Gamma(q)$. Then for any affine connection ∇ on Q, there exists a unique operator $\nabla_{\dot q} : \Gamma(q) \to \Gamma(q)$ (called the *covariant derivative along* q) which agrees with the covariant derivative $\nabla_{\dot q}\tilde W$ for any extension $\tilde W$ of W to Q. A vector field $X \in \Gamma(q)$ is said to be *parallel along* q if $\nabla_{\dot q} X \equiv 0$. The covariant derivative allows to define a particularly important family of smooth curves on Q called *geodesics*, which are defined as the smooth curves γ satisfying $\nabla_{\dot\gamma}\dot\gamma = 0$.

The Riemannian metric induces a unique torsion-free and metric compatible connection called the *Riemannian connection*, or the *Levi-Civita connection* (see Boothby (2003)). Along the rest of the paper, we will assume that ∇ is the Riemannian connection.

2.1 Riemannian geometry and Lie groups

Let G be a Lie group and its Lie algebra $\mathfrak{g}$ be defined as the tangent space to G at the identity, $\mathfrak{g} := T_eG$. Let L_g be the left-translation map $L_g : G \to G$ given by $L_g(h) = gh$, for all $g, h \in G$, which is a left action of G on itself and a diffeomorphism on G. Its tangent map is denoted by $T_hL_g : T_hG \to T_{gh}G$. Let us denote the set of vector fields on a Lie group G by $\mathfrak{X}(G)$. A *left invariant* vector field is an element X of $\mathfrak{X}(G)$ such that $T_hL_g(X(h)) = X(L_g(h)) = X(gh)\ \forall g, h \in G$. We denote the vector space of left-invariant vector fields on G by $\mathfrak{X}_L(G)$. Consider the isomorphism $(\cdot)_L : \mathfrak{g} \to \mathfrak{X}_L(G)$ given by $\xi_L(g) = L_{g*}\xi$, where L_{g*} is the push-forward of the left-translation map L_g, $\xi \in \mathfrak{g}$ and $g \in G$. Note that under this isomorphism we have $\mathfrak{g} \simeq \mathfrak{X}_L(G)$.

Consider an inner-product on the Lie algebra $\mathfrak{g}$ denoted by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. Using the left-translation we define a Riemannian metric on G by the relation $\langle X, Y\rangle := \langle g^{-1}X, g^{-1}Y\rangle_{\mathfrak{g}}$ for all $g \in G$, $X, Y \in T_gG$, which is called left-invariant metric because $\langle gX, gY\rangle = \langle X, Y\rangle$. Let ∇ be the Levi-Civita connection on G related to the above metric. The isomorphism $(\cdot)_L : \mathfrak{g} \to \mathfrak{X}_L(G)$ helps us define an operator $\nabla^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\nabla^{\mathfrak{g}}_\xi \eta := \nabla_{\xi_L}\eta_L(e)$$

for all $\xi, \eta \in \mathfrak{g}$. Although $\nabla^{\mathfrak{g}}$ is not a connection we will refet to it as the Riemannian $\mathfrak{g}$–connection corresponding to ∇.

Lemma 1. *The Riemannian $\mathfrak{g}$–connection satisfies:*

$$\nabla^{\mathfrak{g}}_\xi \eta = \frac{1}{2}\left([\xi,\eta]_{\mathfrak{g}} - \sharp\left[ad^*_\xi \flat(\eta)\right] - \sharp\left[ad^*_\eta \flat(\xi)\right]\right)$$

for all $\xi, \eta \in \mathfrak{g}$.

The geodesics equations on a Lie group equipped with a left-invariant metric might be recast as an equation on the Lie algebra as the well-known result below establishes.

Lemma 2. *Consider a Lie group G with Lie algebra $\mathfrak{g}$ and left-invariant Levi-Civita connection ∇. Let $g : [a,b] \to G$ be a smooth curve and X a smooth vector field along g. Then the following relation holds for all $t \in [a,b]$:*

$$\nabla_{\dot g} X(t) = g(t)\left(\dot\eta(t) + \nabla^{\mathfrak{g}}_\xi \eta(t)\right), \tag{1}$$

where $\xi(t) = g(t)^{-1}\dot g(t)$ and $\eta(t) = g(t)^{-1}X(t)$.

From the previous Lemma, if $g(t)$ is a geodesic with respect to the Levi-Civita connection, then $g\left(\dot\xi + \nabla^{\mathfrak{g}}_\xi\xi\right) = 0$, where $\xi := g^{-1}\dot g$, and we obtain the Euler-Poincare equations for geodesics:

Theorem 1. *Suppose that $g : [a,b] \to G$ is a geodesic, and let $\xi := g^{-1}\dot g$. Then, ξ satisfies*

$$\dot\xi + \nabla^{\mathfrak{g}}_\xi \xi = 0 \tag{2}$$

on $[a,b]$.

3. RIEMANNIAN HOMOGENEOUS SPACES

3.1 Homogeneous spaces

Let G be a connected Lie group. A *homogeneous space* H of G is a smooth manifold on which G acts transitively. Any Lie group is itself a homogeneous space, where the transitive action is given by left-translation (or right-translation).

Suppose that $\Phi : G \times H \to H$ is a transitive left-action, which we denote by $gx := \Phi_g(x)$. It can be shown that for any $x \in H$, we have $G/\mathrm{Stab}(x) \cong H$ as differentiable manifolds, where $\mathrm{Stab}(x) := \{g \in G \mid gx = x\}$ denotes the *stabilizer subgroup* (also called the *isotropy subgroup*) of x, and $G/\mathrm{Stab}(x)$ denotes the space of equivalence classes determined by the equivalence relation $g \sim h$ if and only if $g^{-1}h \in \mathrm{Stab}(x)$. In addition, for any closed Lie subgroup $K \subset G$, the left-action $\Phi : G \times G/K \to G/K$ satisfying $\Phi_g([h]) = [gh]$ for all $g, h \in G$ is transitive, and so G/K is a homogeneous space. Hence, we may assume without loss of generality that $H := G/K$ is a homogeneous space of G for some closed Lie subgroup K.

Let $\pi : G \to H$ be the canonical projection map. We define the *vertical subspace* at $g \in G$ by $\mathrm{Ver}_g := \ker(\pi_*|_g)$, from which we may construct the *vertical bundle* as $VG := \bigsqcup_{g\in G}\{g\} \times \mathrm{Ver}_g$. Given a Riemannian metric $\langle\cdot,\cdot\rangle_G$ on G, we define the *horizontal subspace* at any point $g \in G$ (with respect to $\langle\cdot,\cdot\rangle_G$) as the orthogonal complement of Ver_g. That is, $\mathrm{Hor}_g := \mathrm{Ver}_g^{\perp}$. Similarly, we define the horizontal bundle as $HG := \bigsqcup_{g\in G}\{g\} \times \mathrm{Hor}_g$. Both the vertical and horizontal bundles are vector bundles, and are in fact subbundles of the tangent bundle TG. It is clear that $T_gG = \mathrm{Ver}_g \oplus \mathrm{Hor}_g$ for all $g \in G$, so that the Lie algebra $\mathfrak{g}$ of G admits the decomposition $\mathfrak{g} = \mathfrak{s}\oplus\mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K and $\mathfrak{h} \cong T_{\pi(e)}H$. We denote the orthogonal projections onto the vertical and horizontal subspaces by $\mathcal{V}$ and $\mathcal{H}$.

A section $Z \in \Gamma(HG)$ is called a *horizontal vector field*. That is, $Z \in \Gamma(TG)$ and $Z(g) \in \mathrm{Hor}_g$ for all $g \in G$. A vector field $Y \in \Gamma(TG)$ is said to be π-related to some $X \in \Gamma(TH)$ if $\pi_* Y_g = X_{\pi(g)}$ for all $g \in G$. If in addition $Y \in \Gamma(HG)$, we say that Y is a *horizontal lift* of X. We further define a horizontal lift of a smooth curve $q : [a,b] \to H$ as a smooth curve $\tilde{q} : [a,b] \to G$ such that $\pi \circ \tilde{q} = q$ and $\dot{\tilde{q}}(t)$ is horizontal for all $t \in [a,b]$. We have the following results (see Goodman and Colombo (2024b))

Lemma 3. *Let H be a homogeneous space of G and $X \in \Gamma(TH)$. Then:*

(1) For all $X \in \Gamma(TH)$, there exists a unique horizontal lift $\tilde{X}$ of X. That is, the map $\tilde{\cdot} : \Gamma(TH) \to \Gamma(HG)$ sending $X \mapsto \tilde{X}$ is $\mathbb{R}$-linear and injective.

(2) For all smooth curves $q : [a,b] \to H$ and $q_0 \in \pi^{-1}(\{q(a)\})$, there exists a unique horizontal lift $\tilde{q} : [a,b] \to H$ of q satisfying $\tilde{q}(a) = q_0$, called the horizontal lift of X which is π-related to X.

Lemma 4. *Suppose that $q : [a,b] \to H$ and $\tilde{q} : [a,b] \to G$ is a horizontal lift of q. Then, for any $\tilde{\eta} : [a,b] \to \mathfrak{h}$, there exists a unique $X \in \Gamma(q)$ such that its horizontal lift $\tilde{X}$ along $\tilde{q}$ satisfies $L_{\tilde{q}(t)^{-1}*}\tilde{X}(t) = \tilde{\eta}(t)$ for all $t \in [a,b]$.*

3.2 Riemannian Homogeneous Spaces

Consider a connected Lie group G and a homogeneous space $H = G/K$ of G. Since H is a smooth manifold, it can be equipped with a Riemannian metric. As when discussing Riemannian metrics on Lie groups in Section 2.1, we are interested in those metrics $\langle\cdot,\cdot\rangle_H$ which in some sense preserve the structure of the homogeneous space. In this case, we wish to choose $\langle\cdot,\cdot\rangle_H$ so that the canonical projection map $\pi : G \to H$ is a *Riemannian submersion*. That is, so that $\pi_*|_g$ is a linear isometry between Hor_g and $T_{\pi(g)}H$ for all $g \in G$. In such a case, we call H a *Riemannian homogeneous space*. It is clear that if H is a Riemannian homogeneous space, then $\langle\mathcal{H}(X), \mathcal{H}(Y)\rangle_G = \langle\pi_*X, \pi_*Y\rangle_H$ for all $X, Y \in T_gG, g \in G$. In particular, $\langle\tilde{X}, \tilde{Y}\rangle_G = \langle X, Y\rangle_H$ for all $X, Y \in T_gG, g \in G$. The metric $\langle\cdot,\cdot\rangle_H$ is said to be *G-invariant* if it is invariant under the left-action Φ_g for all $g \in G$. It can be shown that every homogeneous space $H = G/K$ that admits a G-invariant metric is *reductive*. That is, the Lie algebra admits a decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, where $\mathfrak{s}$ is the Lie algebra of K, and $\mathfrak{h}$ satisfies $[\mathfrak{s}, \mathfrak{h}] \subset \mathfrak{h}$. In particular, this implies that $\mathfrak{h} \cong T_{\pi(e)}(G/K)$ as vector spaces.

There is an equivalence between the existence of a G-invariant metric on H and the existence of a left-invariant metric on G for which H is a Riemannian homogeneous space, for more details see Goodman and Colombo (2024b) Denote the Levi-Civita connections on H and G with respect to these metrics by ∇ and $\tilde{\nabla}$ respectively. Also, consider a connection on the horizontal bundle of G, HG, the horizontal connection given by: $\tilde{\nabla}^{\mathcal{H}} : \Gamma(TG) \times \Gamma(HG) \to \Gamma(HG)$,

$$\tilde{\nabla}^{\mathcal{H}}_W Z = \mathcal{H}(\tilde{\nabla}_W Z),$$

for all $W \in \Gamma(TG), Z \in \Gamma(HG)$, which is the projection of the Levi-Civita connection on G onto the horizontal bundle.

Next, we define an operator called the Riemannian $\mathfrak{h}$–connection (see Goodman and Colombo (2024b)) as the bilinear map $\tilde{\nabla}^{\mathfrak{h}}_{\xi}\eta = \tilde{\nabla}^{\mathcal{H}}_{\xi_L}\eta_L(e)$, which is, in essence, the projection of the Riemannian $\mathfrak{g}$–connection, $\tilde{\nabla}^{\mathfrak{g}}$, corresponding to the Levi-Civita connection $\tilde{\nabla}$ on G onto the horizontal subbundle HG, i.e. $\tilde{\nabla}^{\mathfrak{h}}_{\xi}\eta = \mathcal{H}(\tilde{\nabla}^{\mathfrak{g}}_{\xi}\eta)$. Therefore, we obtain the explicit expression

$$\tilde{\nabla}^{\mathfrak{h}}_{\xi}\eta = \frac{1}{2}\mathcal{H}\left([\xi,\eta]_{\mathfrak{g}} - \sharp\left[\mathrm{ad}^*_{\xi}\flat(\eta)\right] - \sharp\left[\mathrm{ad}^*_{\eta}\flat(\xi)\right]\right). \quad (3)$$

Lemma 5. *Consider a Lie group G with Lie algebra $\mathfrak{g}$ and left-invariant Levi-Civita connection ∇. Let $g : [a,b] \to G$ be a horizontal curve and X a smooth horizontal vector field along g. Then the following relation holds for all $t \in [a,b]$:*

$$\tilde{\nabla}^{\mathcal{H}}_{\dot{g}}X(t) = g(t)\left(\dot{\eta}(t) + \tilde{\nabla}^{\mathfrak{h}}_{\xi}\eta(t)\right), \quad (4)$$

where $\xi(t) = g(t)^{-1}\dot{g}(t)$ and $\eta(t) = g(t)^{-1}X(t)$.

From the previous Lemma, if $g(t)$ is a horizontal geodesic with respect to the Levi-Civita connection, then

$$g\left(\dot{\xi} + \tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi\right) = 0,$$

where $\xi := g^{-1}\dot{g}$, and since left-translation is a diffeomorphism, we obtain the equations:

Theorem 2. *Suppose that $g : [a,b] \to G$ is a horizontal geodesic, and let $\xi := g^{-1}\dot{g}$. Then, ξ satisfies*

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi = 0 \tag{5}$$

on $[a,b]$.

Proof. Equation (5) follows directly from Lemma 5. □

Let $V : H \to \mathbb{R}$ be a potential function on the homogeneous space H. Via the projection map π, this potential function generates a potential function on the Lie group G denoted by $\tilde{V} = V \circ \pi$.

Let $g : [a,b] \to G$ be a horizontal curve and a trajectory of the mechanical system with G-invariant Lagrangian function $L : TG \to \mathbb{R}$ of the form kinetic energy minus potential energy and given by $L(g,\dot{g}) = \frac{1}{2}\langle \dot{g},\dot{g}\rangle - \tilde{V}(g)$. Then, the curve g satisfies the equation

$$\tilde{\nabla}^{\mathcal{H}}_{\dot{g}}\dot{g}(t) = -\mathrm{grad}_G\ \tilde{V}(g(t)), \tag{6}$$

where grad_G denotes the gradient with respect to the metric $\langle \cdot,\cdot\rangle$, and we used that $\mathrm{grad}_G\ \tilde{V}$ is a horizontal vector field by construction.

It is not difficult to show that because of G-invariance of the metric and of the function $\tilde{V}$, one might deduce $\mathrm{grad}_G\ \tilde{V}(g) = g\ \mathrm{grad}_G\ \tilde{V}(e)$ and also $\pi_*\left(\mathrm{grad}_G\ \tilde{V}(e)\right) = \mathrm{grad}_H\ V(\pi(e))$. Thus, we deduce that

Theorem 3. *Suppose that $g : [a,b] \to G$ is a horizontal trajectory of the mechanical system with Lagrangian L as defined above and let $\xi := g^{-1}\dot{g}$. Then, ξ satisfies*

$$\dot{\xi} + \tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi = -(\widetilde{grad_H\ V})(\pi(e)). \tag{7}$$

on $[a,b]$.

Proof. The result is a direct consequence of Lemma 5 and of equation (6). □

4. VIRTUAL CONSTRAINTS ON RIEMANNIAN HOMOGENEOUS SPACES

Next, we briefly recall the concept of virtual constraints on a n-dimensional manifold Q. Given an external force $F^0 : TQ \to T^*Q$ and a control force $F : TQ \times U \to T^*Q$ of the form

$$F(q,\dot{q},u) = \sum_{a=1}^{m} u_a f^a(q) \tag{8}$$

where f^a is a one-form on Q with $m < n$, $U \subset \mathbb{R}^m$ the set of controls and $u_a \in \mathbb{R}$ with $1 \leq a \leq m$ the control inputs, consider the associated mechanical control system of the form

$$\nabla_{\dot{q}(t)}\dot{q}(t) = Y^0(q(t),\dot{q}(t)) + u_a(t)Y^a(q(t)), \tag{9}$$

with $Y^0 = \sharp(F^0)$ and $Y^a = \sharp(f^a)$ the corresponding force vector fields.

The distribution $\mathcal{F} \subseteq TQ$ generated by the vector fields $\sharp(f_i)$ is called the *input distribution* associated with the mechanical control system (9). A *virtual constraint* associated with the mechanical control system (9) is a controlled invariant integrable distribution $\mathcal{D} \subseteq TQ$ for that system, that is, there exists a control function $\hat{u} : \mathcal{D} \to \mathbb{R}^m$ such that the solution of the closed-loop system satisfies $\phi_t(\mathcal{D}) \subseteq \mathcal{D}$, where $\phi_t : TQ \to TQ$ denotes its flow.

Provided that $\mathcal{F}$ and $\mathcal{D}$ are complementary (integrable) distributions, there is a unique control law for the system (9) making $\mathcal{D}$ a virtual constraint.

Suppose that H is a homogenous manifold, acted by a Lie group G and $\pi : G \to H$ is the associated projection. Consider a G-invariant control force $F : TH \times U \to T^*H$ of the type (8) with $m < k < n$ where $n = \dim G$, $k = \mathrm{rank}\ H$ and $U \subset \mathbb{R}^m$ the set of controls. Therefore there exists a function $f : T_{\pi(e)}H \times U \to T^*_{\pi(e)}H$ such that

$$F(q,v_q,u) = g\cdot f(g^{-1}q, g^{-1}v_q, u), \quad q \in H,\ v_q \in T_qH,\ u \in U.$$

In particular, the input distribution $\mathcal{F}$ is G-invariant and $\mathcal{F}_q = g\cdot\mathcal{F}_{\pi(e)}$. In addition, suppose that $\mathcal{D}$ is also a G-invariant integrable distribution on H so that $\mathcal{D}_q = g\cdot\mathcal{D}_{\pi(e)}$, where $q \in H$ and $q = g\cdot\pi(e)$.

Using the identification between $\mathfrak{h}$ and $T_{\pi(e)}H$ described in the previous section, there exists a subspace $\mathfrak{d}$ of $\mathfrak{h}$ such that $T_e\pi(\mathfrak{d}) = \mathcal{D}_{\pi(e)}$. Likewise, there exist a subspace $\mathfrak{f} \subseteq \mathfrak{h}$ such that $T_e\pi(\mathfrak{f}) = \mathcal{F}_{\pi(e)}$. These identifications are particularly important for reproducing the trajectories of the Lie algebra on the homogeneous space and vice-versa.

On the Lie algebra $\mathfrak{g}$, consider the controlled mechanical system of the form

$$\dot{g} = g\xi, \quad \dot{\xi} + \tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi + \mathrm{grad}_G\ \tilde{V}(e) = u^a f_a, \tag{10}$$

where $f_a \in \mathfrak{h}$ are the set of $m < k < n$ vectors spanning the control input subspace $\mathfrak{f} = \mathrm{span}\{f_1,\dots,f_m\}$. Note that this system evolves inside the horizontal bundle.

Definition 4. *The subspace of the horizontal subspace $\mathfrak{h} \subseteq \mathfrak{g}$, $\mathfrak{f}$ given above is called the control input subspace associated with the mechanical control system* (10).

Definition 5. *A virtual constraint associated with the mechanical system of type* (10) *is a controlled invariant subspace $\mathfrak{d}$ of $\mathfrak{h}$, that is, there exists a control law making the subspace $\mathfrak{d}$ invariant under the flow of the closed-loop system, i.e. $\xi(0) \in \mathfrak{d}$ and $\xi(t) \in \mathfrak{d}$, $\forall t \geq 0$.*

Theorem 6. *Suppose $\mathfrak{h} = \mathfrak{f} \oplus \mathfrak{d}$. Then there exists a unique control law u^* making $\mathfrak{d}$ a virtual contraint for the controlled mechanical system* (10).

Proof. Let $\dim\mathfrak{d} = d$ and $\dim\mathfrak{f} = m = k - d$. Consider the covectors $\mu^1 \dots, \mu_m \in \mathfrak{h}^*$ spanning the annihiliator subspace of $\mathfrak{d}$. $\xi(t)$ is a curve on $\mathfrak{h}$ satisfying $\xi(t) \in \mathfrak{d}$ for all time if and only if it satifies $\mu^a(\xi(t)) = 0$ for all $a = 1,\dots,m$. Differentiating this equation and supposing that $\xi(t)$ is a solution of the closed loop system (10) for an appropriate choice of control law u, we have that

$$-\mu^a\left(\tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi + \mathrm{grad}_G\ \tilde{V}(e)\right) + u^b\mu^a(f_b) = 0.$$

Since $\tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi + \mathrm{grad}_G\ \tilde{V}(e) \in \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{f}\oplus\mathfrak{d}$, there is a unique way to decompose this vector as the sum

$$\tilde{\nabla}^{\mathfrak{h}}_{\xi}\xi + \mathrm{grad}_G\ \tilde{V}(e) = \eta(t) + \tau^b(t)f_b,$$

with $\eta \in \mathfrak{d}$. In addition, note that the coefficients τ^b may be regarded as functions on $\mathfrak{h}$. In fact, its definition is associated with the projection to $\mathfrak{f}$ together with the choice of $\{f_b\}$ as a basis for $\mathfrak{f}$. Therefore, $\mu^a(\dot{\xi}(t)) = 0$ if and only if

$$(\tau^b - u^b)\mu^a(f_b) = 0.$$

Since $\mu^a(f_b)$ is an invertible matrix, we conclude that $\tau^b = u^b$ proving existence and uniqueness of a control law making $\mathfrak{d}$ a virtual constraint. □

By construction, if $(g(t), \xi(t))$ is a trajectory of the mechanical system (10), then $g(t)$ is a horizontal curve and $q(t) = \pi(g(t))$ is the trajectory of the mechanical system

$$\nabla^H_{\dot{q}(t)}\dot{q}(t) = -\text{grad}_H\ V(q(t)) + u_a(t)Y^a(q(t)).$$

In addition, if $\xi(t) \in \mathfrak{d}$ for all t, then $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. Then, by uniqueness of the control law making the integrable distribution $\mathcal{D}$ a virtual constraint, we have that the control law $u^* \in U$ given in Theorem 6 is also the unique control law making $\mathcal{D}$ control invariant.

5. AN EXAMPLE

Consider a spherical pendulum with varying length. Let the spherical body be of mass m and the length of be given by the variable ℓ. Its configuration space is $\mathbb{S}^2\times]0,+\infty[$ and we are interested in a control system of the type

$$\nabla_{\dot{q}(t)}\dot{q}(t) = -\text{grad}\ V(q(t)) + u(t)Y(q(t))$$

where ∇ is the Levi-Civita connection on $\mathbb{S}^2\times]0,+\infty[$ relative to the metric induced by the euclidean metric on $\mathbb{R}^4$ and V is a potential function. We would like to find a control law forcing the pendulum to move with constant length. For that matter, we assume that $\mathcal{D} = T\mathbb{S}^2$ and the input distribution determined is spanned by the vector field $\{Y\}$ which is a vector tangent to the vertical space of the projection onto the first component $\text{pr}_1 : \mathbb{S}^2\times]0,+\infty[\to \mathbb{S}^2$, $(q,\ell) \mapsto q$.

Finding an explicit expression for this control law can be cumbersome since one has to deal with a local expression of the Levi-Civita connection and the expression is limited to the region where the coordinates are properly defined. In the case of the sphere, there will be just one point not covered by the coordinates but the issue might be even more dramatic for other more complex manifolds. Working with homogeneous manifolds solves some of these issues since the goal with this strategy is to work in a linear space: a subspace of the Lie algebra of the acting Lie group. In addition, from the engineering point of view, many times the controllers are themselves an element of the Lie algebra: angular velocity, torque, etc. Therefore, in the following, we will tackle the control problem making use of the homogeneous structure of the configuration manifold.

Homogeneous structure of $\mathbb{S}^2\times]0,+\infty[$. The configuration space $\mathbb{S}^2\times]0,+\infty[$ can be identified with the homogeneous space $H = G/K$ for $G = \text{SO}(3)\times]0,+\infty[$ and $K \subset G$ be the Lie subgroup isomorphic to $SO(2)$ whose elements are given by $(k,1) \in G$ with $k = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\theta \in \mathbb{S}^1$. From now on, to ease the notation we will use the real set $\mathbb{R}$ instead of $]0,+\infty[$. G has the left multiplication from $SO(3)$ extended to $SO(3)\times\mathbb{R}$, i.e., $(S,r)\cdot(R,\ell) = (SR, r\ell)$ with $S \in SO(3)$, $r \in \mathbb{R}$. The action of G on $\mathbb{S}^2\times\mathbb{R}$, denoted by $\Phi : G\times\mathbb{S}^2\times\mathbb{R} \to \mathbb{S}^2\times\mathbb{R}$ and given by $\Phi_{(S,r)}(q,\ell) = (Sq, r\ell)$ is transitive and therefore H is in fact an homogeneous space of G. For $e_3 = [0\ 0\ 1]^T$ we get that $K = \text{Stab}((e_3, 1))$ the stabilizer subgroup of the action Φ. The projection map $\pi : G \to \mathbb{S}^2 \times \mathbb{R}$ is given by $\pi(R,\ell) = (Re_3, \ell)$. By abuse of notation, we will denote the projection $SO(3) \to \mathbb{S}^2$ also by π, i.e., $\pi(R) = Re_3, R \in \text{SO}(3)$.

The Lie algebra $\mathfrak{so}(3)\times\mathbb{R}$. Using the hat map, we identify $\mathfrak{so}(3) \cong \mathbb{R}^3$. Consider the orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ of $\mathfrak{so}(3)$, where e_1, e_2, e_3 is the standard basis on $\mathbb{R}^3$ so, we have $\pi_*\hat{e}_1 = -e_2, \pi_*\hat{e}_2 = e_1$ and $\pi_*\hat{e}_3 = 0$.

Riemmannian homogeneous structure. Let us equip the Lie group SO(3) with a left-invariant metric defined by the inner product on $\mathfrak{so}(3)$ given by $\langle\hat{\Omega}_1, \hat{\Omega}_2\rangle_{\text{SO}(3)} = \Omega_1^T\Omega_2$ for all $\Omega_1, \Omega_2 \in \mathbb{R}^3$. We have that $\mathfrak{s} = \ker(\pi_*|_I) = \text{span}\{\hat{e}_3\}$ and we can define $\mathfrak{h} = \mathfrak{s}^\perp$. Note that $\mathfrak{h} = \text{span}\{\hat{e}_1, \hat{e}_2\}$. We define an inner product on $T_{e_3}\mathbb{S}^2$ via the relation $\langle X, Y\rangle_{T_{e_3}\mathbb{S}^2} := \langle\pi_*^{-1}X, \pi_*^{-1}Y\rangle_{\text{SO}(3)}$ for all $X, Y \in T_{e_3}\mathbb{S}^2$. Following [Goodman and Colombo (2024a)] we have that $\langle\cdot,\cdot\rangle_{T_{e_3}\mathbb{S}^2}$ is the standard Euclidean metric with respect to the basis $\{e_1, e_2\}$. We extend this inner product to an SO(3)-invariant Riemannian metric on $\mathbb{S}^2$ by left-action given by $\langle X, Y\rangle_{\mathbb{S}^2} = \langle R^TX, R^TY\rangle_{T_{e_3}\mathbb{S}^2} = R^TX\cdot R^TY = X\cdot Y$ for all $X, Y \in T_q\mathbb{S}^2, R \in \text{SO}(3)$ such that $\pi(R) = q$. The metric on $G = \text{SO}(3)\times\mathbb{R}$ is simply given by $\langle(\hat{\Omega}_1, r_1), (\hat{\Omega}_2, r_2)\rangle_G = \langle\hat{\Omega}_1, \hat{\Omega}_2\rangle_{\text{SO}(3)} + r_1r_2$. The sharp map $\sharp : T^*G \to TG$ is given by $\sharp(\mu, r) = (\mu^T, r)$. The adjoint operation of $\mathfrak{g} = \mathfrak{so}(3)\times\mathbb{R}$ to itself is given by $\text{ad} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$,

$$\text{ad}_\xi\eta = (\text{ad}_{\hat{\Omega}_1}\hat{\Omega}_2, 0) = ([\hat{\Omega}_1, \hat{\Omega}_2], 0) = (\widehat{\Omega_1\times\Omega_2}, 0)$$

where $\text{ad}_{\hat{\Omega}_1}\hat{\Omega}_2$ is the adjoint operator on $\mathfrak{so}(3)$, $\xi = (\hat{\Omega}_1, r_1)$ and $\eta = (\hat{\Omega}_2, r_2)$.

Since $\mathfrak{s} = \text{span}\{\hat{e}_3\}$ the vertical space is defined by $Ver_g = \text{span}\{g\hat{e}_3\}$ and the horizontal space is defined as $Hor_g = \text{span}\{g\hat{e}_1, g\hat{e}_2\}$ for $g \in \text{SO}(3)$. Thus, for G we get $Ver_g \times \{0\}$ and $Hor_g \times \mathbb{R}$ the vertical and horizontal spaces respectively. The horizontal projection can be calculated by $\mathcal{H}(\hat{\Omega}, r) = (\widehat{\Omega\times e_3}, r)$.

The $\mathfrak{h}$*-connection on the Lie algebra.* Hence, from (3) we have that

$$\tilde{\nabla}^{\mathfrak{h}}_\xi\eta = \frac{1}{2}\mathcal{H}(\widehat{\Omega_1\times\Omega_2}, 0) = \frac{1}{2}\left((\widehat{\Omega_1\times\Omega_2)\times e_3}, 0\right),$$

where $\xi = (\hat{\Omega}_1, r_1)$ and $\eta = (\hat{\Omega}_2, r_2)$.

For a horizonal curve $\bar{R} : [a,b] \to G$ and a smooth horizontal vector field X along $\bar{R}$ from Lemma 5 we have

$$\tilde{\nabla}^{\mathcal{H}}_{\dot{\bar{R}}}X(t) = \bar{R}(t)\left(\dot{\eta} + \tilde{\nabla}^{\mathfrak{h}}_\xi\eta(t)\right) = (R, r)\left(\dot{\hat{\Omega}}_2 + \frac{1}{2}(\widehat{\Omega_1\times\Omega_2})e_3, 0\right),$$

where $\eta = \bar{R}^{-1}X$, $\xi = \bar{R}^{-1}\dot{\bar{R}}$ and $\bar{R} = (R, r)$. In particular, if $\bar{R}$ is a horizontal geodesic then from equation (5) $\xi = (\hat{\Omega}, \ell)$ satisfies

$$\dot{\Omega} = 0$$
$$\dot{\ell} = 0.$$

The control problem on $\mathfrak{h}$. Now, consider the gravitational potential function as $V : \mathbb{S}^2\times\mathbb{R} \to \mathbb{R}$ given by $V(q,\ell) = \ell e_3^Tq$. We can pull back this function to G using the projection $\tilde{V} : G \to \mathbb{R}$ is defined as $\tilde{V}(R,\ell) =$

$\ell e_3^T R e_3$. The gradient with respect to G of this function is $\text{grad}_G \tilde{V}(I,1) = (0,1)$.

We would like to make the subspace $\mathfrak{d} = \text{span}\{\hat{e}_1, \hat{e}_2\} \subseteq \mathfrak{h}\times\mathbb{R}$ control invariant. With that in mind, let $\mathfrak{f} = \text{span}\{f\}$, where $f = (0,1) \in \mathfrak{so}(3) \times \mathbb{R}$. Thus, we look for a control law making the system

$$\dot{\Omega} = 0$$
$$\dot{\ell} = -1 + u.$$

control invariant. It is clear that this is only possible if $u \equiv 1$.

6. CONCLUSIONS AND FUTURE WORK

We introduced the notion of a virtual constraints on Riemannian homogeneous spaces in a geometric framework, which is a generalization of the classical controlled invariant distribution setting and we show the existence and uniqueness of a control law preserving the invariant subspace. We illustrated the theory with the example of the damped spherical pendulum. In a future paper, we will study in-depth the relation between the control law on the horizontal space $\mathfrak{h}$ and the control law on the homogenous space H.

In addition, we aim to extend this notion of virtual constraint on Riemannian homogeneous spaces to the class of nonholonomic virtual constraints as in our recent works Simoes et al. (2023), Stratoglou et al. (2023a), Stratoglou et al. (2023b) for linear, affine and nonlinear nonholonomic virtual constraints. A typical example of these constraints on Riemannian homogeneous spaces is a rolling ball on a table, with and without rotational motion of the table (see Bloch (2003) for instance).

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