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# CYCLIC MAYA DIAGRAMS AND RATIONAL SOLUTIONS OF HIGHER ORDER PAINLEVÉ SYSTEMS

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**ABSTRACT.** This paper focuses on the construction of rational solutions for the  $A_{2n}$ -Painlevé system, also called the Noumi-Yamada system, which are considered the higher order generalizations of  $P_{IV}$ . In this even case, we introduce a method to construct the rational solutions based on cyclic dressing chains of Schrödinger operators with potentials in the class of rational extensions of the harmonic oscillator. Each potential in the chain can be indexed by a single Maya diagram and expressed in terms of a Wronskian determinant whose entries are Hermite polynomials. We introduce the notion of cyclic Maya diagrams and we characterize them for any possible period, using the concepts of genus and interlacing. The resulting classes of solutions can be expressed in terms of special polynomials that generalize the families of generalized Hermite, generalized Okamoto and Umemura polynomials, showing that they are particular cases of a larger family.

**Keywords.** Painlevé equations, Noumi-Yamada systems, rational solutions, Darboux dressing chains, Maya diagrams, Wronskian determinants, Hermite polynomials.

## 1. INTRODUCTION

The set of six nonlinear second order Painlevé equations  $P_I, \dots, P_{VI}$  have been the focus of intense study from many different angles in the past century [23, 43]. Their defining property is that their solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation. The Painlevé equations, whose solutions are called Painlevé transcendents, are now considered to be the nonlinear analogues of special functions, cf. [20, 23, 30]. Although, in general, these functions are transcendental in the sense that they cannot be expressed in terms of previously known functions, the Painlevé equations, except the first, also possess special families of solutions that can be expressed via rational functions, algebraic functions or the classical special functions, such as Airy, Bessel, parabolic cylinder, Whittaker or hypergeometric functions, for special values of the parameters, see, for example, [23, 43] and the references therein.

Rational solutions of the second Painlevé equation ( $P_{II}$ )

$$(1.1) \quad u'' = 2u^3 + zu + \alpha,$$

with  $\alpha$  an arbitrary constant and  $' \equiv d/dz$ , were studied by Yablonskii [78] and Vorob'ev [76], in terms of a special class of polynomials, now known as the *Yablonskii–Vorob'ev polynomials*  $Q_n(z)$ , which are polynomials of degree  $\frac{1}{2}n(n+1)$ . Clarkson and Mansfield [26] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and observed that these roots have a very regular, approximately triangular structure; the term “approximate” is used since the patterns are not exact triangles as the roots lie on arcs rather than straight lines. Bertola and Bothner [11] and Buckingham and Miller [15, 16] studied the Yablonskii–Vorob'ev polynomials  $Q_n(z)$  in the limit as  $n \rightarrow \infty$  and showed that the roots lie in a triangular region with elliptic sides which meet with interior angle  $\frac{2}{5}\pi$ . Further Buckingham and Miller [15, 16] show that in the limit as  $n \rightarrow \infty$ , the rational solution of  $P_{II}$  tends to the *tritonquée solution* of the first Painlevé equation ( $P_I$ )

$$(1.2) \quad u'' = 6u^2 + z.$$

Okamoto [62] obtained special polynomials associated with some of the rational solutions of the fourth Painlevé equation ( $P_{IV}$ )

$$(1.3) \quad u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u},$$

with  $\alpha$  and  $\beta$  constants, which are analogous to the Yablonskii–Vorob’ev polynomials. Noumi and Yamada [55] generalized Okamoto’s results and expressed all rational solutions of  $P_{IV}$  in terms of two types of special polynomials, now known as the *generalized Hermite polynomials*  $H_{m,n}(z)$  and *generalized Okamoto polynomials*  $Q_{m,n}(z)$ , both of which are determinants of sequences of Hermite polynomials. The structure of the roots of these polynomials is studied in [21], where it is shown that the roots of the generalized Hermite polynomials have an approximate rectangular structure and the roots of the generalized Okamoto polynomials have a combination of approximate rectangular and triangular structures. Recent studies on the asymptotic distribution of zeros of certain generalised Hermite polynomials  $H_{m,n}(z)$  as  $m, n \rightarrow \infty$  are given in [13, 50, 51] and of the Okamoto polynomials in [56]. Buckingham [13] also obtained an explicit characterization of the boundary curve in the case of generalized Hermite polynomials.

Umemura [73] derived special polynomials associated with rational solutions of the third Painlevé equation ( $P_{III}$ ) and the fifth Painlevé equation ( $P_V$ ) which are determinants of sequences of associated Laguerre polynomials.

The special polynomials associated with rational solutions of the Painlevé equations arise in several applications:

- (i) the Yablonskii–Vorob’ev polynomials arise in the transition behaviour for the semi-classical sine-Gordon equation [14], in boundary value problems [8], in moving boundary problems [65, 67], and in symmetry reductions of a Korteweg capillarity system [68] and cold plasma physics [69];
- (ii) the generalized Hermite polynomials arise as multiple integrals in random matrix theory [31], in supersymmetric quantum mechanics [9, 10, 49, 57], in the description of vortex dynamics with quadrupole background flow [24], and as coefficients of recurrence relations for orthogonal polynomials [19, 25, 74];
- (iii) the generalized Okamoto polynomials arise in supersymmetric quantum mechanics [49] and generate rational-oscillatory solutions of the de-focusing nonlinear Schrödinger equation [22];
- (iv) the Umemura polynomials arise as multivortex solutions of the complex sine-Gordon equation [7, 64], and in Multiple-Input-Multiple-Output wireless communication systems [18].

A very successful approach in the study of rational solutions to Painlevé equations has been through the geometric methods developed by the Japanese school, most notably by Noumi and Yamada [54]. The core idea is to write the scalar equations as a system of first order nonlinear equations. For instance, Noumi and Yamada [54] showed that  $P_{IV}$  (1.3) is equivalent to the following system of three first order equations

$$(1.4) \quad \begin{aligned} f'_0 + f_0(f_1 - f_2) &= \alpha_0, \\ f'_1 + f_1(f_2 - f_0) &= \alpha_1, \\ f'_2 + f_2(f_0 - f_1) &= \alpha_2, \end{aligned}$$

with  $' \equiv d/dz$  and  $\alpha_j$ ,  $j = 0, 1, 2$  constants, subject to the normalization conditions

$$(1.5) \quad f_0 + f_1 + f_2 = z, \quad \alpha_0 + \alpha_1 + \alpha_2 = 1.$$

Once this equivalence is shown, it is clear that the symmetric form of  $P_{IV}$  (1.4), sometimes referred to as  $sP_{IV}$ , is easier to analyse. In particular, Noumi and Yamada [55] showed that the system (1.4) possesses a symmetry group of Bäcklund transformations acting on the tuple of solutions and parameters  $(f_0, f_1, f_2 | \alpha_0, \alpha_1, \alpha_2)$ . This symmetry group is the affine Weyl group  $A_2^{(1)}$ , generated by

the operators  $\{\pi, \mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2\}$  whose action on the tuple  $(f_0, f_1, f_2 | \alpha_0, \alpha_1, \alpha_2)$  is given by:

$$(1.6) \quad \begin{aligned} \mathbf{s}_k(f_j) &= f_j - \frac{\alpha_k \delta_{k+1,j}}{f_k} + \frac{\alpha_k \delta_{k-1,j}}{f_k}, \\ \mathbf{s}_k(\alpha_j) &= \alpha_j - 2\alpha_j \delta_{k,j} + \alpha_k (\delta_{k+1,j} + \delta_{k-1,j}), \\ \pi(f_j) &= f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1} \end{aligned}$$

where  $\delta_{k,j}$  is the Kronecker delta and  $j, k = 0, 1, 2 \pmod{3}$ . The technique to generate rational solutions is to first identify a number of very simple rational *seed solutions*, and then successively apply the Bäcklund transformations (1.6) to generate families of rational solutions.

This is a beautiful approach which makes use of the hidden group theoretic structure of transformations of the equations, but the solutions built by dressing seed solutions are not very explicit, in the sense that one needs to iterate a number of Bäcklund transformations (1.6) on the functions and parameters in order to obtain the desired solutions. Questions such as determining the number of zeros or poles of a given solution constructed in this manner seem very difficult to address. For this reason, alternative representations of the rational solutions have also been investigated, most notably the determinantal representations [44, 45], and representations in terms of Schur polynomials, [55] and universal characters [72].

The system of first order equations (1.4) admits a natural generalization to any number of equations, and it is known as the  $A_N$ -Painlevé or the Noumi-Yamada system. The  $A_{2n}$ -Painlevé system is considerably simpler (for reasons that will be explained later), and it is the one we will focus on this paper. In this case, the system has the form:

$$(1.7) \quad f'_i + f_i \left( \sum_{j=1}^n f_{i+2j-1} - \sum_{j=1}^n f_{i+2j} \right) = \alpha_i, \quad i = 0, \dots, 2n \pmod{2n+1}$$

subject to the normalization conditions

$$(1.8) \quad f_0 + \dots + f_{2n} = z, \quad \alpha_0 + \dots + \alpha_{2n} = 1.$$

The symmetry group of this higher order system is the affine Weyl group  $A_{2n}^{(1)}$ , acting by Bäcklund transformations as in (1.6). The system passes the Painlevé-Kowalevskaya test, [75], and it is believed to possess the Painlevé property. It is thus considered a proper higher order generalization of sP<sub>IV</sub> (1.4), which corresponds to the special case  $n = 1$ .

The next higher order system belonging to this hierarchy is the  $A_4$ -Painlevé system, that has been studied by Filipuk and Clarkson [29], who provide several classes of rational solutions via an explicit Wronskian representation, and by Matsuda [53], who uses the classical approach to identify the set of parameters that lead to rational solutions. However, a complete classification and explicit description of the rational solutions of  $A_{2n}$ -Painlevé for  $n \geq 2$  is, to the best of our knowledge, still not available in the literature.

Of particular interest are the special polynomials associated with these rational solutions, whose zeros and poles structure shows extremely regular patterns in the complex plane, and have received a considerable amount of study, as mentioned above. We will show that all these special polynomials are only particular cases of a larger one.

Our approach for describing rational solutions to the Noumi-Yamada system makes no use of symmetry groups of Bäcklund transformations, vertex operators, Hirota bilinear equations, etc. Instead, we will adopt the approach of Darboux dressing chains introduced by the Russian school [3, 75], which has received comparatively less attention in connection to Painlevé systems, and the recent advances in the theory of exceptional polynomials [35, 38, 39]. Having said this, it would be a very interesting development to establish a dictionary between the symmetry group approach and the one presented in this paper.

The paper is organized as follows: in Section 2 we introduce the equations for a dressing chain of Darboux transformations of Schrödinger operators and prove that they are equivalent to the  $A_{2n}$ -Painlevé system. These results are well known 3 but recalling them is useful to fix notation and make the paper self contained. In Section 3 we explore the class of dressing chains built on rational extensions of the harmonic oscillator, which can be indexed by Maya diagrams. We introduce the key notion of *cyclic Maya diagrams* and reformulate the problem of classifying rational solutions of the  $A_{2n}$ -Painlevé system as that of classifying  $(2n+1)$ -cyclic Maya diagrams. In Section 4 we introduce the notion of genus and interlacing for Maya diagrams which allows us to achieve a complete classification of  $p$ -cyclic Maya diagrams for any period  $p$ . In Section 5, we focus on the  $A_4$ -Painlevé system to give the class of rational solutions using the representation developed in the previous sections. Finally, we show some plots of the roots of these special solutions in the complex plane.

The purpose of this paper is to illustrate a new construction method and an explicit representation of rational solutions to higher order Painlevé systems. We conjecture that this construction includes all possible rational solutions to the system. A proof of this fact requires new arguments than the ones developed in this paper, and remains for now an open question.

Even cyclic dressing chains provide higher order extensions of  $P_V$ . The situation for this even cyclic case corresponding to the  $A_{2n+1}$ -Painlevé systems is considerably harder. Construction methods similar to the ones described here are available, but the class of dressing chains is larger, and it includes rational extensions of both the harmonic and the isotonic oscillator 41,42, thus described by universal characters (or pairs of Maya diagrams). Some rational solutions have been given by Tsuda 72 but the full classification for this case is still an open question.

## 2. DARBOUX DRESSING CHAINS

The theory of dressing chains, or sequences of Schrödinger operators connected by Darboux transformations was developed by Adler 3, and Veselov and Shabat 75. The connection between dressing chains and Painlevé equations was already shown in 3 and it has been exploited by some authors 10,47-49,52,70-72,77. This section follows mostly the early works of Adler, Veselov and Shabat.

Consider the following sequence of Schrödinger operators

$$(2.1) \quad L_i = -D_z^2 + U_i, \quad D_z = \frac{d}{dz}, \quad U_i = U_i(z), \quad i \in \mathbb{Z}$$

where each operator is related to the next by a Darboux transformation, i.e. by the following factorization

$$(2.2) \quad \begin{aligned} L_i &= (D_z + w_i)(-D_z + w_i) + \lambda_i, & w_i &= w_i(z), \\ L_{i+1} &= (-D_z + w_i)(D_z + w_i) + \lambda_i. \end{aligned}$$

It follows that the functions  $w_i$  satisfy the Riccati equations

$$(2.3) \quad w'_i + w_i^2 = U_i - \lambda_i, \quad -w'_i + w_i^2 = U_{i+1} - \lambda_i.$$

Equivalently,  $w_i$  are the log-derivatives of  $\psi_i$ , the seed function of the Darboux transformation that maps  $L_i$  to  $L_{i+1}$

$$(2.4) \quad L_i \psi_i = \lambda_i \psi_i, \quad \text{where } w_i = \frac{\psi'_i}{\psi_i}.$$

Using (2.1) and (2.2), the potentials of the dressing chain are related by

$$(2.5) \quad U_{i+1} = U_i - 2w'_i,$$

$$(2.6) \quad U_{i+n} = U_i - 2(w'_i + \dots + w'_{i+n-1}), \quad n \geq 2$$

If we eliminate the potentials in (2.3) and set

$$(2.7) \quad a_i = \lambda_i - \lambda_{i+1}$$

the following chain of coupled equations is obtained

$$(w_i + w_{i+1})' + w_{i+1}^2 - w_i^2 = a_i, \quad i \in \mathbb{Z}$$

Before continuing, note that this infinite chain of equations has the evident reversal symmetry

$$(2.8) \quad w_i \mapsto -w_{-i}, \quad a_i \mapsto -a_{-i}.$$

This infinite chain of equations closes and becomes a finite dimensional system of ordinary differential equations if a cyclic condition is imposed on the potentials of the chain

$$(2.9) \quad U_{i+p} = U_i + \Delta, \quad i \in \mathbb{Z}$$

for some  $p \in \mathbb{N}$  and  $\Delta \in \mathbb{C}$ . If this holds, then necessarily  $w_{i+p} = w_i$ ,  $a_{i+p} = a_i$ , and

$$(2.10) \quad \Delta = -(a_0 + \dots + a_{p-1}).$$

**Definition 2.1.** A  $p$ -cyclic Darboux dressing chain (or factorization chain) with shift  $\Delta$  is a sequence of  $p$  functions  $w_0, \dots, w_{p-1}$  and complex numbers  $a_0, \dots, a_{p-1}$  that satisfy the following coupled system of  $p$  Riccati-like ordinary differential equations

$$(2.11) \quad (w_i + w_{i+1})' + w_{i+1}^2 - w_i^2 = a_i, \quad i = 0, 1, \dots, p-1 \pmod{p}$$

subject to the condition (2.10).

Note that transformation

$$(2.12) \quad w_i \mapsto -w_{-i}, \quad a_i \mapsto -a_{-i}, \quad \Delta \mapsto -\Delta$$

projects the reversal symmetry to the finite-dimensional system (2.11). Moreover, for  $j = 0, 1, \dots, p-1$  we also have the cyclic symmetry

$$w_i \mapsto w_{i+j}, \quad a_i \mapsto a_{i+j}, \quad \Delta \mapsto \Delta \quad i = 0, \dots, p-1 \pmod{p}$$

In the classification of solutions to (2.11) it will be convenient to regard two solutions related by a reversal symmetry or by a cyclic permutation as being equivalent.

Adding the  $p$  equations (2.11) we immediately obtain a first integral of the system

$$\sum_{j=0}^{p-1} w_j = \frac{1}{2}z \sum_{j=0}^{p-1} a_j = -\frac{1}{2}\Delta z.$$

**Remark 2.2.** We assume throughout this paper that  $\Delta \neq 0$ , which is the only case for which the dressing chain in Definition 2.1 leads to solutions of the  $A_N$ -Painlevé system with normalization (1.8) (and thus to higher order generalizations of  $P_{\text{IV}}$ ). In the  $\Delta = 0$  case, the dressing chain (2.11) defines a completely integrable system whose general solution can be expressed in terms of elliptic and theta functions, [75], the corresponding Schrödinger operators belonging to the class of finite-gap potentials. The rational solutions in this class can be expressed in terms of Burchall-Chaundy [17] (or Adler-Moser [1]) polynomials, which can be obtained by confluent Darboux-Crum transformations of the free potential at zero energy, [27].

In the  $\Delta \neq 0$  case, the  $A_{2n}$ -Painlevé system (1.7) and the cyclic dressing chain (2.11) are related by the following proposition.

**Proposition 2.3.** *If the tuple of functions and complex numbers  $(w_0, \dots, w_{2n} | a_0, \dots, a_{2n})$  satisfies a  $(2n+1)$ -cyclic Darboux dressing chain with shift  $\Delta \neq 0$  as per Definition 2.1, then the tuple  $(f_0, \dots, f_{2n} | \alpha_0, \dots, \alpha_{2n})$  with*

$$(2.13) \quad f_i(z) = c(w_i + w_{i+1})(cz), \quad i = 0, \dots, 2n \pmod{2n+1},$$

$$(2.14) \quad \alpha_i = c^2 a_i,$$

$$(2.15) \quad c^2 = -\frac{1}{\Delta}$$

solves the  $A_{2n}$ -Painlevé system (1.7) with normalization (1.8).

*Proof.* The linear transformation

$$(2.16) \quad f_i = w_i + w_{i+1}, \quad i = 0, \dots, 2n \pmod{2n+1}$$

is invertible (only in the odd case  $p = 2n+1$ ), the inverse transformation being

$$(2.17) \quad w_i = \frac{1}{2} \sum_{j=0}^{2n} (-1)^j f_{i+j}, \quad i = 0, \dots, 2n \pmod{2n+1}$$

They imply the relations

$$(2.18) \quad w_{i+1} - w_i = \sum_{j=0}^{2n-1} (-1)^j f_{i+j+1}, \quad i = 0, \dots, 2n \pmod{2n+1}.$$

Inserting (2.16) and (2.18) into the equations of the cyclic dressing chain (2.11) leads to the  $A_{2n}$ -Painlevé system (1.7). For any constant  $c \in \mathbb{C}$ , the scaling transformation

$$f_i \mapsto cf_i, \quad z \mapsto cz, \quad \alpha_i \mapsto c^2 \alpha_i$$

preserves the form of the equations (1.7). The choice  $c^2 = -\frac{1}{\Delta}$  ensures that the normalization (1.8) always holds, for dressing chains with different shifts  $\Delta$ .  $\square$

**Remark 2.4.**  $(2n)$ -cyclic dressing chains and  $A_{2n-1}$ -Painlevé systems are also related, but the mapping is given by a rational rather than a linear function. A full treatment of this even cyclic case (which includes  $P_V$  and its higher order hierarchy) is considerably harder and shall be treated elsewhere.

The problem now becomes that of finding and classifying cyclic dressing chains, i.e. Schrödinger operators and sequences of Darboux transformations that reproduce the initial potential up to an additive shift  $\Delta$  after a fixed given number of transformations.

The theory of exceptional polynomials is intimately related with families of Schrödinger operators connected by Darboux transformations [32, 40]. Constructing cyclic dressing chains on this class of potentials becomes a feasible task, and knowledge of the effect of rational Darboux transformations on the potentials suggests that the only family of potentials to be considered in the case of odd cyclic dressing chains are the rational extensions of the harmonic oscillator [34], which are exactly solvable potentials whose eigenfunctions are expressible in terms of exceptional Hermite polynomials.

Each potential in this class can be indexed by a finite set of integers (specifying the sequence of Darboux transformations applied on the harmonic oscillator that lead to the potential), or equivalently by a Maya diagram, which becomes a very useful representation to capture a notion of equivalence and relations of the type (2.9).

As mentioned before, the fact that all rational odd cyclic dressing chains (and equivalently rational solutions to the  $A_{2n}$ -Painlevé system) must *necessarily* belong to this class remains an open question. We conjecture that this is indeed the case, and no rational solutions other than the ones described in the following sections exist.

## 3. CYCLIC MAYA DIAGRAMS AND RATIONAL EXTENSIONS OF THE HARMONIC OSCILLATOR

In this Section we construct odd cyclic dressing chains on potentials belonging to the class of rational extensions of the harmonic oscillator. Every such potential is represented by a Maya diagram, a rational Darboux transformation acting on this class will be a flip operation on a Maya diagram and cyclic Darboux chains correspond to cyclic Maya diagrams. With this representation, the main problem of constructing rational cyclic Darboux chains becomes purely algebraic and combinatorial.

Following Noumi [54], we define a Maya diagram in the following manner.

**Definition 3.1.** A Maya diagram is a set of integers  $M \subset \mathbb{Z}$  that contains a finite number of positive integers, and excludes a finite number of negative integers. We will use  $\mathcal{M}$  to denote the set of all Maya diagrams.

**Definition 3.2.** Let  $m_1 > m_2 > \dots$  be the elements of a Maya diagram  $M$  arranged in decreasing order. By assumption, there exists a unique integer  $s_M \in \mathbb{Z}$  such that  $m_i = -i + s_M$  for all  $i$  sufficiently large. We define  $s_M$  to be the index of  $M$ .

We visualize a Maya diagram as a horizontally extended sequence of  $\blacksquare$  and  $\square$  symbols with the filled symbol  $\blacksquare$  in position  $i$  indicating membership  $i \in M$ . The defining assumption now manifests as the condition that a Maya diagram begins with an infinite filled  $\blacksquare$  segment and terminates with an infinite empty  $\square$  segment.

**Definition 3.3.** Let  $M$  be a Maya diagram, and

$$M_- = \{-m - 1 : m \notin M, m < 0\}, \quad M_+ = \{m : m \in M, m \geq 0\}.$$

Let  $s_1 > s_2 > \dots > s_p$  and  $t_1 > t_2 > \dots > t_q$  be the elements of  $M_-$  and  $M_+$  arranged in descending order.

We define the *Frobenius symbol* of  $M$  to be the double list  $(s_1, \dots, s_p \mid t_q, \dots, t_1)$ .

It is not hard to show that  $s_M = q - p$  is the index of  $M$ . The classical Frobenius symbol [4, 5, 63] corresponds to the zero index case where  $q = p$ . If  $M$  is a Maya diagram, then for any  $k \in \mathbb{Z}$  so is

$$M + k = \{m + k : m \in M\}.$$

The behaviour of the index  $s_M$  under translation of  $k$  is given by

$$(3.1) \quad M' = M + k \quad \Rightarrow \quad s_{M'} = s_M + k.$$

We will refer to an equivalence class of Maya diagrams related by such shifts as an *unlabelled Maya diagram*. One can visualize the passage from an unlabelled to a labelled Maya diagram as the choice of placement of the origin.

A Maya diagram  $M \subset \mathbb{Z}$  is said to be in standard form if  $p = 0$  and  $t_q > 0$ . Visually, a Maya diagram in standard form has only filled boxes  $\blacksquare$  to the left of the origin and one empty box  $\square$  just to the right of the origin. Every unlabelled Maya diagram permits a unique placement of the origin so as to obtain a Maya diagram in standard form.

In [35] it was shown that to every Maya diagram we can associate a polynomial called a Hermite pseudo-Wronskian.

**Definition 3.4.** Let  $M$  be a Maya diagram and  $(s_1, \dots, s_r \mid t_q, \dots, t_1)$  its corresponding Frobenius symbol. Define the polynomial

$$(3.2) \quad H_M(z) = \exp(-rz^2) \text{Wr}[\exp(z^2)\tilde{H}_{s_1}, \dots, \exp(z^2)\tilde{H}_{s_r}, H_{t_q}, \dots, H_{t_1}],$$

where  $\text{Wr}$  denotes the Wronskian determinant of the indicated functions, and

$$(3.3) \quad \tilde{H}_n(z) = i^{-n} H_n(iz)$$

is the  $n^{\text{th}}$  degree conjugate Hermite polynomial.

The polynomial nature of  $H_M(z)$  becomes evident once we represent it using a slightly different determinant.

**Proposition 3.5.** *The Wronskian  $H_M(z)$  admits the following alternative determinantal representation*

$$(3.4) \quad H_M(z) = \begin{vmatrix} \tilde{H}_{s_1} & \tilde{H}_{s_1+1} & \cdots & \tilde{H}_{s_1+r+q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{H}_{s_r} & \tilde{H}_{s_r+1} & \cdots & \tilde{H}_{s_r+r+q-1} \\ H_{t_q} & D_z H_{t_q} & \cdots & D_z^{r+q-1} H_{t_q} \\ \vdots & \vdots & \ddots & \vdots \\ H_{t_1} & D_z H_{t_1} & \cdots & D_z^{r+q-1} H_{t_1} \end{vmatrix}$$

The term Hermite pseudo-Wronskian was coined in [35] because (3.4) is a mix of a Casoratian and a Wronskian determinant. For all Maya diagrams in the same equivalence class, their associated Hermite pseudo-Wronskians enjoy a very simple relation.

**Proposition 3.6** ([35] Theorem 1). *Let  $\hat{H}_M(z)$  be the normalized pseudo-Wronskian*

$$(3.5) \quad \hat{H}_M(z) = \frac{(-1)^{r+q} H_M(z)}{\prod_{1 \leq i < j \leq r} (2s_j - 2s_i) \prod_{1 \leq i < j \leq q} (2t_i - 2t_j)}.$$

Then for any Maya diagram  $M$  and  $k \in \mathbb{Z}$  we have

$$(3.6) \quad \hat{H}_M(z) = \hat{H}_{M+k}(z).$$

The remarkable aspect of equation (3.6) is that the identity involves determinants of different sizes. Note that the statement of this proposition is slightly different than the original result proved in [35], due to the introduction of *normalized* pseudo-Wronskians (3.5) to achieve strict equality in (3.6) rather than just equality up to a multiplicative constant. As mentioned above, every unlabelled Maya diagram contains a Maya diagram in standard form, and its associated Hermite pseudo-Wronskian (3.2) is just an ordinary Wronskian determinant whose entries are Hermite polynomials. This will not be in general the smallest determinant in the equivalence class. The procedure to find the smallest equivalent determinant was given in [35].

Due to Proposition 3.6, we could restrict the analysis without loss of generality to Maya diagrams in standard form and Wronskians of Hermite polynomials, but we will employ the general notation as it brings conceptual clarity to the description of cyclic Maya diagrams.

We will now introduce and study a class of potentials for Schrödinger operators that will be used as building blocks for cyclic dressing chains. Rational extensions of classic potentials have been studied in a number of papers [6, 41, 42, 48, 60] and their set of eigenfunctions are expressible in terms of exceptional orthogonal polynomials.

**Definition 3.7.** A rational extension of the harmonic oscillator is a potential of the form

$$U(z) = z^2 + \frac{a(z)}{b(z)},$$

with  $a(x)$ ,  $b(x)$  polynomials with  $\deg a \leq \deg b$ , that is *exactly solvable by polynomials*. This means that for all but finitely many  $n \in \mathbb{N}$ , the operator  $L = -D^2 + U(z)$  has formal eigenfunctions of the form

$$\psi_n = \mu(z)y_n(z),$$

where  $\mu(z)$  is a fixed function and where  $y_n(z)$  are polynomials of degree  $n$ .

If  $b(z)$  has no real zeros, then  $L$  is a Sturm-Liouville operator on  $\mathbb{R}$  with quasi-polynomial eigenfunctions. Exact solvability by polynomials is a very stringent property, which is equivalent to trivial monodromy [27, 58]. In fact, the next Proposition proved in [34] states that rational extensions of the harmonic oscillator can be put in one to one correspondence with Maya diagrams.

**Proposition 3.8** ([34] Theorem 1.1). *Let  $M \subset \mathbb{Z}$  be a Maya diagram. Define*

$$(3.7) \quad U_M(z) = z^2 - 2D_z^2 \log H_M(z) + 2s_M,$$

where  $H_M(z)$  is the corresponding pseudo-Wronskian (3.2)–(3.4), and  $s_M \in \mathbb{Z}$  is the index of  $M$ . Up to an additive constant, every rational extension of the harmonic oscillator takes the form (3.7).

The class of Schrödinger operators with potentials that are rational extensions of the harmonic oscillator is invariant under a certain class of rational Darboux transformations, which we now describe.

**Definition 3.9.** We define the flip at position  $m \in \mathbb{Z}$  to be the involution  $\phi_m : \mathcal{M} \rightarrow \mathcal{M}$  defined by

$$(3.8) \quad \phi_m : M \mapsto \begin{cases} M \cup \{m\}, & \text{if } m \notin M, \\ M \setminus \{m\}, & \text{if } m \in M. \end{cases} \quad M \in \mathcal{M}.$$

In the first case, we say that  $\phi_m$  acts on  $M$  by a state-deleting transformation ( $\square \rightarrow \blacksquare$ ). In the second case, we say that  $\phi_m$  acts by a state-adding transformation ( $\blacksquare \rightarrow \square$ ).

It can be shown that every quasi-rational eigenfunction [36, 37] of  $L = -D_z^2 + U_M(z)$  has the form

$$(3.9) \quad \psi_{M,m} = \exp\left(\frac{1}{2}\varepsilon z^2\right) \frac{H_{\phi_m(M)}(z)}{H_M(z)}, \quad m \in \mathbb{Z},$$

with

$$\varepsilon = \begin{cases} -1, & \text{if } m \notin M, \\ +1, & \text{if } m \in M, \end{cases}$$

Explicitly, we have

$$(3.10) \quad L\psi_{M,m} = (2m+1)\psi_{M,m}, \quad m \in \mathbb{Z}.$$

**Remark 3.10.** The seed eigenfunctions (3.9) include the true eigenfunctions of  $L$  plus another set of formal non square-integrable eigenfunctions, sometimes known in the physics literature as *virtual states*, [59, 61]. For a correct spectral theoretic interpretation one needs to ensure that the potential  $U_M$  is regular, i.e. that  $H_M(z)$  has no zeros in  $\mathbb{R}$ . The set of Maya diagrams for which  $H_M(z)$  has no real zeros was characterized (in a more general setting) independently by Krein [46] and Adler [2], while the number of real zeros for  $H_M$  was given in [33]. However, for the purpose of this paper it is convenient to stay within a purely formal setting and keep the whole class of potentials  $U_M$ , regardless of whether they have real poles or not.

The relation between dressing chains of Darboux transformations for the class of operators (3.7) and flip operations on Maya diagrams is made explicit by the following proposition.

**Proposition 3.11.** *Two Maya diagrams  $M, M'$  are related by a flip (3.8) if and only if their associated rational extensions  $U_M, U_{M'}$ , see (3.7), are connected by a Darboux transformation (2.5).*

*Proof.* Suppose that  $m \notin M$  and that  $M' = M \cup \{m\}$  is a state-deleting flip transformation of  $M$ . The seed function for the factorization is  $\psi_{M,m}$  defined in (3.9). Set

$$(3.11) \quad w_{M,m}(z) = \frac{\psi'_{M,m}(z)}{\psi_{M,m}(z)} = -z + \frac{H'_{M'}(z)}{H_{M'}(z)} - \frac{H'_M(z)}{H_M(z)}.$$

Since

$$s_{M'} = s_M + 1,$$

by (3.7), we have

$$(3.12) \quad \frac{1}{2}[U_{M'}(z) - U_M(z)] = 1 + \frac{d}{dz} \left( \frac{H'_M(z)}{H_M(z)} - \frac{H'_{M'}(z)}{H_{M'}(z)} \right) = -w'_{M,m}(z),$$

so that (2.5) holds. Conversely, suppose that  $M$  and  $M'$  are such that (3.12) holds for some  $w = w(z)$ . If we define

$$w(z) = \frac{\psi'(z)}{\psi(z)}, \quad \psi(z) = \exp(-\frac{1}{2}z^2) \frac{H_{M'}(z)}{H_M(z)},$$

then  $\psi$  must be a quasi-rational seed function for  $U_M$  and it follows by (3.9) of Proposition 3.8 that  $M' = M \cup \{m\}$  for some  $m \notin M$ . The corresponding result for state-adding Darboux transformations is done in a similar way.  $\square$

We see thus that the class of rational extensions of the harmonic oscillator is indexed by Maya diagrams, and that the Darboux transformations that preserve this class can be described by flip operations on Maya diagrams. It now becomes feasible to characterize cyclic dressing chains built on this class of potentials.

**Definition 3.12.** For  $p \in \mathbb{N}$  let  $\mathcal{Z}_p$  denote the set of all subsets of  $\mathbb{Z}$  having cardinality  $p$ . For  $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_p\} \in \mathcal{Z}_p$  we now define  $\phi_{\boldsymbol{\mu}}$  to be the multi-flip

$$(3.13) \quad \phi_{\boldsymbol{\mu}} = \phi_{\mu_1} \circ \dots \circ \phi_{\mu_p}.$$

We are now ready to introduce the basic concept of this section.

**Definition 3.13.** We say that  $M$  is  $p$ -cyclic with shift  $k$ , or  $(p, k)$  cyclic, if there exists a  $\boldsymbol{\mu} \in \mathcal{Z}_p$  such that

$$(3.14) \quad \phi_{\boldsymbol{\mu}}(M) = M + k.$$

We will say that  $M$  is  $p$ -cyclic if it is  $(p, k)$  cyclic for some  $k \in \mathbb{Z}$ .

**Proposition 3.14.** For Maya diagrams  $M, M' \in \mathcal{M}$ , we define the set  $\Upsilon(M, M')$  as the symmetric difference between  $M$  and  $M'$ :

$$(3.15) \quad \Upsilon(M, M') = (M \setminus M') \cup (M' \setminus M).$$

Then the multi-flip  $\phi_{\boldsymbol{\mu}}$  where  $\boldsymbol{\mu} = \Upsilon(M, M')$  is the unique multi-flip such that  $M' = \phi_{\boldsymbol{\mu}}(M)$  and  $M = \phi_{\boldsymbol{\mu}}(M')$ .

As an immediate corollary, we have the following.

**Proposition 3.15.** Let  $k$  be a non-zero integer. Every Maya diagram  $M \in \mathcal{M}$  is  $(p, k)$  cyclic where  $p$  is the cardinality of  $\boldsymbol{\mu} = \Upsilon(M, M + k)$ .

We are now able to establish the link between cyclic Maya diagrams and cyclic dressing chains composed of rational extensions of the harmonic oscillator.

**Proposition 3.16.** Let  $M \in \mathcal{M}$  be a Maya diagram,  $k$  a non-zero integer, and  $p$  the cardinality of  $\boldsymbol{\mu} = \Upsilon(M, M + k)$ . Let  $\boldsymbol{\mu} = \{\mu_0, \dots, \mu_{p-1}\}$  be an arbitrary enumeration of  $\boldsymbol{\mu}$  and set

$$(3.16) \quad M_0 = M, \quad M_{i+1} = \phi_{\mu_i}(M_i), \quad i = 0, 1, \dots, p-1$$

so that  $M_p = M_0 + k$  by construction. Set

$$(3.17) \quad w_i(z) = s_i z + \frac{H'_{M_{i+1}}(z)}{H_{M_{i+1}}(z)} - \frac{H'_{M_i}(z)}{H_{M_i}(z)}, \quad i = 0, \dots, p-1.$$

$$(3.18) \quad a_i = 2(\mu_i - \mu_{i+1}),$$

where

$$(3.19) \quad s_i = \begin{cases} -1, & \text{if } \mu_i \notin M, \\ +1, & \text{if } \mu_i \in M, \end{cases}$$

and

$$\mu_p = \mu_0 + k.$$

Then,  $(w_0, \dots, w_{p-1}; a_0, \dots, a_{p-1})$  constitutes a rational solution to the  $p$ -cyclic dressing chain (2.11) with shift  $\Delta = 2k$ .

*Proof.* The result follows from the structure of the seed eigenfunctions (3.9) with eigenvalues given by (3.10), after applying (2.4) and (2.7). The sign of  $s_i$  indicates whether the  $(i+1)$ -th step of the chain that takes  $L_i$  to  $L_{i+1}$  is a state-adding (+1) or state-deleting (-1) transformation.  $\square$

The remaining part of the construction is to classify cyclic Maya diagrams for any given (odd) period, which we tackle next. Under the correspondence described by Proposition 3.16 the reversal symmetry (2.12) manifests as the transformation

$$(M_0, \dots, M_p) \mapsto (M_p, \dots, M_0), \quad (\mu_1, \dots, \mu_p) \mapsto (\mu_p, \dots, \mu_1), \quad k \mapsto -k.$$

In light of the above remark, there is no loss of generality if we restrict our attention to cyclic Maya diagrams with a positive shift  $k > 0$ .

#### 4. CLASSIFICATION OF CYCLIC MAYA DIAGRAMS

In this section we introduce the key concepts of *genus* and *interlacing* to achieve a full classification of cyclic Maya diagrams.

For  $\beta \in \mathcal{Z}_{2g+1}$  define the Maya diagram

$$(4.1) \quad \Xi(\beta) = (-\infty, \beta_0) \cup [\beta_1, \beta_2) \cup \dots \cup [\beta_{2g-1}, \beta_{2g})$$

where

$$[m, n) = \{j \in \mathbb{Z}: m \leq j < n\}$$

and where  $\beta_0 < \beta_1 < \dots < \beta_{2g}$  is the strictly increasing enumeration of  $\beta$ .

**Proposition 4.1.** *Every Maya diagram  $M \in \mathcal{M}$  has a unique representation of the form  $M = \Xi(\beta)$  where  $\beta$  is a set of integers of odd cardinality  $2g + 1$ .*

**Definition 4.2.** We call the integer  $g \geq 0$  the genus of  $M = \Xi(\beta)$  and  $(\beta_0, \beta_1, \dots, \beta_{2g})$  the block coordinates of  $M$ .

**Proposition 4.3.** *Let  $M = \Xi(\beta)$  be a Maya diagram specified by its block coordinates. We then have*

$$\beta = \Upsilon(M, M + 1).$$

*Proof.* Observe that

$$M + 1 = (-\infty, \beta_0] \cup (\beta_1, \beta_2] \cup \dots \cup (\beta_{2g-1}, \beta_{2g}],$$

where

$$(m, n] = \{j \in \mathbb{Z}: m < j \leq n\}.$$

It follows that

$$\begin{aligned} (M + 1) \setminus M &= \{\beta_0, \dots, \beta_{2g}\} \\ M \setminus (M + 1) &= \{\beta_1, \dots, \beta_{2g-1}\}. \end{aligned}$$

The desired conclusion follows immediately.  $\square$

Let  $\mathcal{M}_g$  denote the set of Maya diagrams of genus  $g$ . The above discussion may be summarized by saying that the mapping (4.1) defines a bijection  $\Xi : \mathcal{Z}_{2g+1} \rightarrow \mathcal{M}_g$ , and that the block coordinates are precisely the flip sites required for a translation  $M \mapsto M + 1$ .

**Remark 4.4.** To motivate Definition 4.2, it is perhaps more illustrative to understand the visual meaning of the genus of  $M$ , see Figure 4.1. After removal of the initial infinite  $\blacksquare$  segment and the trailing infinite  $\square$  segment, a Maya diagram consists of alternating empty  $\square$  and filled  $\blacksquare$  segments of variable length. The genus  $g$  counts the number of such pairs. The even block coordinates  $\beta_{2i}$  indicate the starting positions of the empty segments, and the odd block coordinates  $\beta_{2i+1}$  indicated the starting positions of the filled segments. Also, note that  $M$  is in standard form if and only if  $\beta_0 = 0$ .

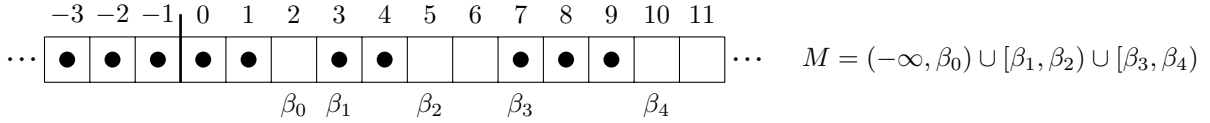


FIGURE 4.1. Block coordinates  $(\beta_0, \dots, \beta_4) = (2, 3, 5, 7, 10)$  of a genus 2 Maya diagrams. Note that the genus is both the number of finite-size empty blocks and the number of finite-size filled blocks.

The next concept we need to introduce is the interlacing and modular decomposition.

**Definition 4.5.** Fix a  $k \in \mathbb{N}$  and let  $M^{(0)}, M^{(1)}, \dots, M^{(k-1)} \subset \mathbb{Z}$  be sets of integers. We define the interlacing of these to be the set

$$(4.2) \quad \Theta \left( M^{(0)}, M^{(1)}, \dots, M^{(k-1)} \right) = \bigcup_{i=0}^{k-1} (kM^{(i)} + i),$$

where

$$kM + j = \{km + j : m \in M\}, \quad M \subset \mathbb{Z}.$$

Dually, given a set of integers  $M \subset \mathbb{Z}$  and a  $k \in \mathbb{N}$  define the sets

$$M^{(i)} = \{m \in \mathbb{Z} : km + i \in M\}, \quad i = 0, 1, \dots, k-1.$$

We will call the  $k$ -tuple of sets  $(M^{(0)}, M^{(1)}, \dots, M^{(k-1)})$  the  $k$ -modular decomposition of  $M$ .

The following result follows directly from the above definitions.

**Proposition 4.6.** *We have  $M = \Theta(M^{(0)}, M^{(1)}, \dots, M^{(k-1)})$  if and only if  $(M^{(0)}, M^{(1)}, \dots, M^{(k-1)})$  is the  $k$ -modular decomposition of  $M$ .*

Even though the above operations of interlacing and modular decomposition apply to general sets, they have a well defined restriction to Maya diagrams. Indeed, it is not hard to check that if  $M = \Theta(M^{(0)}, M^{(1)}, \dots, M^{(k-1)})$  and  $M$  is a Maya diagram, then  $M^{(0)}, M^{(1)}, \dots, M^{(k-1)}$  are also Maya diagrams. Conversely, if the latter are all Maya diagrams, then so is  $M$ . Another important case concerns the interlacing of finite sets. The definition (4.2) implies directly that if  $\mu^{(i)} \in \mathcal{Z}_{p_i}$ ,  $i = 0, 1, \dots, k-1$  then

$$\mu = \Theta \left( \mu^{(0)}, \dots, \mu^{(k-1)} \right)$$

is a finite set of cardinality  $p = p_0 + \dots + p_{k-1}$ .

Visually, each of the  $k$  Maya diagrams is dilated by a factor of  $k$ , shifted by one unit with respect to the previous one and superimposed, so the interlaced Maya diagram incorporates the information from  $M^{(0)}, \dots, M^{(k-1)}$  in  $k$  different modular classes. An example can be seen in Figure 4.2. In other words,

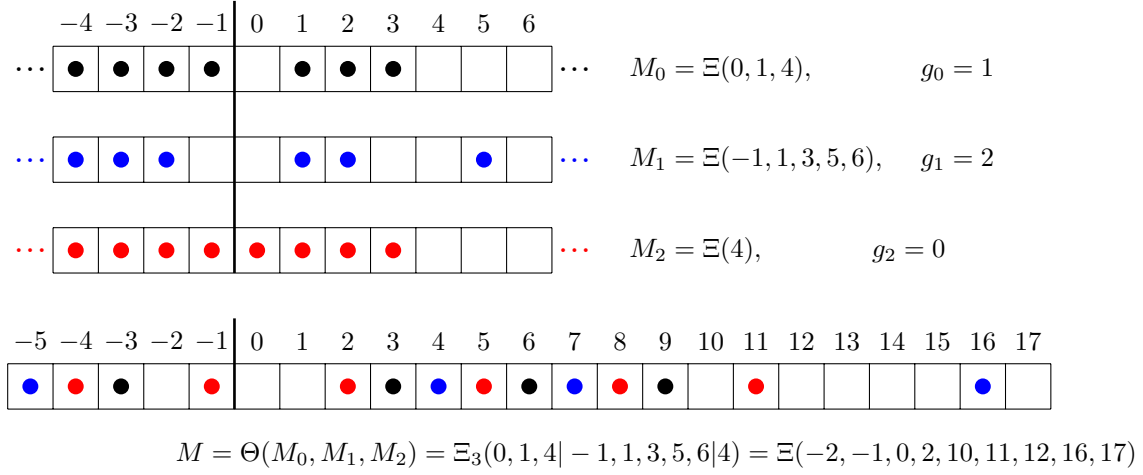


FIGURE 4.2. Interlacing of three Maya diagrams with genus 1, 2 and 0 with block coordinates and 3-block coordinates for the interlaced Maya diagram.

the interlaced Maya diagram is built by copying sequentially a filled or empty box as determined by each of the  $k$  Maya diagrams.

**Remark 4.7.** Modular decomposition of Maya diagrams has been considered previously by Noumi in his book [54] (see Proposition 7.12), although in a different context: that of studying the effect of Bäcklund transformations on the Maya diagrams. In the present context of dressing chains and rational solutions, Tsuda [72] has also employed the notation in (4.2) for the interlacing of  $N$  genus-0 Maya diagrams, which correspond to  $N$ -reduced partitions. This particular family of rational solutions correspond to the signature class  $(1, 1, \dots, 1)$  with the highest shift  $k = 2n + 1$  (see Section 5 below), i.e. the generalization of Okamoto polynomials.

Equipped with these notions of genus and interlacing, we are now ready to state the main result for the classification of cyclic Maya diagrams.

**Theorem 4.8.** *Let  $M = \Theta(M^{(0)}, M^{(1)}, \dots, M^{(k-1)})$  be the  $k$ -modular decomposition of a given Maya diagram  $M$ . Let  $g_i$  be the genus of  $M^{(i)}$ ,  $i = 0, 1, \dots, k-1$ . Then,  $M$  is  $(p, k)$ -cyclic where*

$$(4.3) \quad p = p_0 + p_1 + \dots + p_{k-1}, \quad p_i = 2g_i + 1.$$

*Proof.* Let  $\beta^{(i)} = \Upsilon(M^{(i)}, M^{(i)} + 1) \in \mathcal{Z}_{p_i}$  be the block coordinates of  $M^{(i)}$ ,  $i = 0, 1, \dots, k-1$ . Consider the interlacing  $\mu = \Theta(\beta^{(0)}, \dots, \beta^{(k-1)})$ . From Proposition 4.3 we have that,

$$\phi_{\beta^{(i)}}(M^{(i)}) = M^{(i)} + 1.$$

so it follows that

$$\begin{aligned}
 \phi_{\mu}(M) &= \phi_{\Theta(\beta^{(0)}, \dots, \beta^{(k-1)})} \Theta(M^{(0)}, \dots, M^{(k-1)}) \\
 &= \Theta(\phi_{\beta^{(0)}}(M^{(0)}), \dots, \phi_{\beta^{(k-1)}}(M^{(k-1)})) \\
 &= \Theta(M^{(0)} + 1, \dots, M^{(k-1)} + 1) \\
 &= \Theta(M^{(0)}, \dots, M^{(k-1)}) + k \\
 &= M + k.
 \end{aligned}$$

Therefore,  $M$  is  $(p, k)$  cyclic where the value of  $p$  agrees with (4.3).  $\square$

Theorem 4.8 sets the way to classify cyclic Maya diagrams for any given period  $p$ .

**Corollary 4.9.** *For a fixed period  $p \in \mathbb{N}$ , there exist  $p$ -cyclic Maya diagrams with shifts  $k = p, p - 2, \dots, \lfloor p/2 \rfloor$ , and no other positive shifts are possible.*

**Remark 4.10.** The highest shift  $k = p$  corresponds to the interlacing of  $p$  trivial (genus 0) Maya diagrams.

We now introduce a combinatorial system for describing rational solutions of  $p$ -cyclic factorization chains. First, we require a suitably generalized notion of block coordinates suitable for describing  $p$ -cyclic Maya diagrams.

**Definition 4.11.** Let  $M = \Theta(M^{(0)}, \dots, M^{(k-1)})$  be a  $k$ -modular decomposition of a  $(p, k)$  cyclic Maya diagram. For  $i = 0, 1, \dots, k-1$  let  $\beta^{(i)} = (\beta_0^{(i)}, \dots, \beta_{p_i-1}^{(i)})$  be the block coordinates of  $M^{(i)}$  enumerated in increasing order. In light of the fact that

$$M = \Theta(\Xi(\beta^{(0)}), \dots, \Xi(\beta^{(k-1)})),$$

we will refer to the concatenated sequence

$$\begin{aligned} (\beta_0, \beta_1, \dots, \beta_{p-1}) &= (\beta^{(0)} | \beta^{(1)} | \dots | \beta^{(k-1)}) \\ &= (\beta_0^{(0)}, \dots, \beta_{p_0-1}^{(0)} | \beta_0^{(1)}, \dots, \beta_{p_1-1}^{(1)} | \dots | \beta_0^{(k-1)}, \dots, \beta_{p_{k-1}-1}^{(k-1)}) \end{aligned}$$

as the  $k$ -block coordinates of  $M$ . Formally, the correspondence between  $k$ -block coordinates and Maya diagram is described by the mapping

$$\Xi_k: \mathcal{Z}_{2g_0+1} \times \dots \times \mathcal{Z}_{2g_{k-1}+1} \rightarrow \mathcal{M}$$

with action

$$\Xi_k: (\beta^{(0)} | \beta^{(1)} | \dots | \beta^{(k-1)}) \mapsto \Theta(\Xi(\beta^{(0)}), \dots, \Xi(\beta^{(k-1)}))$$

**Definition 4.12.** Fix a  $k \in \mathbb{N}$ . For  $m \in \mathbb{Z}$  let  $[m]_k \in \{0, 1, \dots, k-1\}$  denote the residue class of  $m$  modulo division by  $k$ . For  $m, n \in \mathbb{Z}$  say that  $m \preceq_k n$  if and only if

$$[m]_k < [n]_k, \quad \text{or} \quad [m]_k = [n]_k \text{ and } m \leq n.$$

In this way, the transitive, reflexive relation  $\preceq_k$  forms a total order on  $\mathbb{Z}$ .

**Proposition 4.13.** *Let  $M$  be a  $(p, k)$  cyclic Maya diagram. There exists a unique  $p$ -tuple of integers  $(\mu_0, \dots, \mu_{p-1})$  strictly ordered relative to  $\preceq_k$  such that*

$$(4.4) \quad \phi_{\mu}(M) = M + k$$

*Proof.* Let  $(\beta_0, \dots, \beta_{p-1}) = (\beta^{(0)} | \beta^{(1)} | \dots | \beta^{(k-1)})$  be the  $k$ -block coordinates of  $M$ . Set

$$\mu = \Theta(\beta^{(0)}, \dots, \beta^{(k-1)})$$

so that (4.4) holds by the proof to Theorem 4.8. The desired enumeration of  $\mu$  is given by

$$(k\beta_0, \dots, k\beta_{p-1}) + (0^{p_0}, 1^{p_1}, \dots, (k-1)^{p_{k-1}})$$

where the exponents indicate repetition. Explicitly,  $(\mu_0, \dots, \mu_{p-1})$  is given by

$$(k\beta_0^{(0)}, \dots, k\beta_{p_0-1}^{(0)}, k\beta_0^{(1)} + 1, \dots, k\beta_{p_1-1}^{(1)} + 1, \dots, k\beta_0^{(k-1)} + k - 1, \dots, k\beta_{p_{k-1}-1}^{(k-1)} + k - 1).$$

$\square$

**Definition 4.14.** In light of (4.4) we will refer to the just defined tuple  $(\mu_0, \mu_1, \dots, \mu_{p-1})$  as the  $k$ -canonical flip sequence of  $M$  and refer to the tuple  $(p_0, p_1, \dots, p_{k-1})$  as the  $k$ -signature of  $M$ .

By Proposition 3.16 a rational solution of the  $p$ -cyclic dressing chain requires a  $(p, k)$  cyclic Maya diagram, and an additional item data, namely a fixed ordering of the canonical flip sequence. We will specify such ordering as

$$\boldsymbol{\mu}_\pi = (\mu_{\pi_0}, \dots, \mu_{\pi_{p-1}})$$

where  $\pi = (\pi_0, \dots, \pi_{p-1})$  is a permutation of  $(0, 1, \dots, p-1)$ . With this notation, the chain of Maya diagrams described in Proposition 3.16 is generated as

$$(4.5) \quad M_0 = M, \quad M_{i+1} = \phi_{\mu_{\pi_i}}(M_i), \quad i = 0, 1, \dots, p-1.$$

**Remark 4.15.** Using a translation it is possible to normalize  $M$  so that  $\mu_0 = 0$ . Using a cyclic permutation and it is possible to normalize  $\pi$  so that  $\pi_p = 0$ . The net effect of these two normalizations is to ensure that  $M_0, M_1, \dots, M_{p-1}$  have standard form.

**Remark 4.16.** In the discussion so far we have imposed the hypothesis that the sequence of flips that produces a translation  $M \mapsto M + k$  does not contain any repetitions. However, in order to obtain a full classification of rational solutions, it will be necessary to account for degenerate chains which include multiple flips at the same site.

To that end it is necessary to modify Definition 3.12 to allow  $\boldsymbol{\mu}$  to be a multi-set<sup>1</sup>, and to allow  $\mu_0, \mu_1, \dots, \mu_{p-1}$  in (4.1) to be merely a non-decreasing sequence. This has the effect of permitting  $\square$  and  $\blacksquare$  segments of zero length wherever  $\mu_{i+1} = \mu_i$ . The  $\Xi$ -image of such a non-decreasing sequence is not necessarily a Maya diagram of genus  $g$ , but rather a Maya diagram whose genus is bounded above by  $g$ .

It is no longer possible to assert that there is a unique  $\boldsymbol{\mu}$  such that  $\phi_{\boldsymbol{\mu}}(M) = M + k$ , because it is possible to augment the non-degenerate  $\boldsymbol{\mu} = \Upsilon(M, M + k)$  with an arbitrary number of pairs of flips at the same site to arrive at a degenerate  $\boldsymbol{\mu}'$  such that  $\phi_{\boldsymbol{\mu}'}(M) = M + k$  also. The rest of the theory remains unchanged.

## 5. RATIONAL SOLUTIONS OF $A_4$ -PAINLEVÉ

In this section we will put together all the results derived above in order to describe an effective way of labelling and constructing all the rational solutions to the  $A_{2k}$ -Painlevé system based on cyclic dressing chains of rational extensions of the harmonic oscillator. We conjecture that the construction described below covers all rational solutions to such systems. As an illustrative example, we describe all rational solutions to the  $A_4$ -Painlevé system, and we furnish examples in each signature class.

For odd  $p$ , in order to specify a Maya  $p$ -cycle, or equivalently a rational solution of a  $p$ -cyclic dressing chain, we need to specify three items of data:

- (i) a signature sequence  $(p_0, \dots, p_{k-1})$  consisting of odd positive integers that sum to  $p$ . This sequence determines the genus of the  $k$  interlaced Maya diagrams that give rise to a  $(p, k)$ -cyclic Maya diagram  $M$ . The possible values of  $k$  are given by Corollary 4.9
- (ii) Once the signature is fixed, we need to specify the  $k$ -block coordinates

$$(\beta_0, \dots, \beta_{p-1}) = (\boldsymbol{\beta}^{(0)} | \dots | \boldsymbol{\beta}^{(k-1)})$$

where  $\boldsymbol{\beta}^{(i)} = (\beta_0^{(i)}, \dots, \beta_{p_i}^{(i)})$  are the block coordinates that define each of the interlaced Maya diagrams  $M^{(i)}$ . These two items of data specify uniquely a  $(p, k)$ -cyclic Maya diagram  $M$ , and a canonical flip sequence  $\boldsymbol{\mu} = (\beta_0, \dots, \beta_{p-1})$ . The next item specifies a given  $p$ -cycle that contains  $M$ .

- (iii) Once the  $k$ -block coordinates and canonical flip sequence  $\boldsymbol{\mu}$  are fixed, we still have the freedom to choose a permutation  $\pi \in S_p$  of  $(0, 1, \dots, p-1)$  that specifies the actual flip sequence  $\boldsymbol{\mu}_\pi$ , i.e. the order in which the flips in the canonical flip sequence are applied to build the Maya  $p$ -cycle.

<sup>1</sup>A multi-set is generalization of the concept of a set that allows for multiple instances for each of its elements.

For any signature of a Maya  $p$ -cycle, we need to specify the  $p$  integers in the canonical flip sequence, but following Remark 4.15, we can get rid of translation invariance by setting  $\mu_0 = \beta_0^{(0)} = 0$ , leaving only  $p - 1$  free integers. Moreover, we can restrict ourselves to permutations such that  $\pi_p = 0$  in order to remove the invariance under cyclic permutations. The remaining number of degrees of freedom is  $p - 1$ , which (perhaps not surprisingly) coincides with the number of generators of the symmetry group  $A_{p-1}^{(1)}$ . This is a strong indication that the class described above captures a generic orbit of a seed solution under the action of the symmetry group.

We now illustrate the general theory by describing the rational solutions of the  $A_4^{(1)}$ -Painlevé system, whose equations are given by

$$(5.1) \quad \begin{aligned} f'_0 + f_0(f_1 - f_2 + f_3 - f_4) &= \alpha_0, \\ f'_1 + f_1(f_2 - f_3 + f_4 - f_0) &= \alpha_1, \\ f'_2 + f_2(f_3 - f_4 + f_0 - f_1) &= \alpha_2, \\ f'_3 + f_3(f_4 - f_0 + f_1 - f_2) &= \alpha_3, \\ f'_4 + f_4(f_0 - f_1 + f_2 - f_3) &= \alpha_4, \end{aligned}$$

with normalization conditions

$$f_0 + f_1 + f_2 + f_3 + f_4 = z, \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1.$$

This system has the “seed solutions”

$$\begin{aligned} (f_0, f_1, f_2, f_3, f_4) &= (z, 0, 0, 0, 0), & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (1, 0, 0, 0, 0), \\ (f_0, f_1, f_2, f_3, f_4) &= (\tfrac{1}{3}z, \tfrac{1}{3}z, \tfrac{1}{3}z, 0, 0), & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0), \\ (f_0, f_1, f_2, f_3, f_4) &= (\tfrac{1}{5}z, \tfrac{1}{5}z, \tfrac{1}{5}z, \tfrac{1}{5}z, \tfrac{1}{5}z), & (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}), \end{aligned}$$

and permutations thereof.

**Theorem 5.1.** *Rational solutions of the  $A_4^{(1)}$ -Painlevé system (5.1) correspond to chains of 5-cyclic Maya diagrams belonging to one of the following signature classes:*

$$(5), (3, 1, 1), (1, 3, 1), (1, 1, 3), (1, 1, 1, 1, 1).$$

*With the normalization  $\pi_4 = 0$  and  $\mu_0 = 0$ , each rational solution may be uniquely labelled by one of the above signatures, a 4-tuple of arbitrary non-negative integers  $(n_1, n_2, n_3, n_4)$ , and a permutation  $(\pi_0, \pi_1, \pi_2, \pi_3)$  of  $(1, 2, 3, 4)$ . For each of the above signatures, the corresponding  $k$ -block coordinates of the initial 5-cyclic Maya diagram are then given by*

$$\begin{array}{lll} k = 1 & (5) & (0, n_1, n_1 + n_2, n_1 + n_2 + n_3, n_1 + n_2 + n_3 + n_4) \\ k = 3 & (3, 1, 1) & (0, n_1, n_1 + n_2 | n_3 | n_4) \\ k = 3 & (1, 3, 1) & (0 | n_1, n_1 + n_2, n_1 + n_2 + n_3 | n_4) \\ k = 3 & (1, 1, 3) & (0 | n_1 | n_2, n_2 + n_3, n_2 + n_3 + n_4) \\ k = 5 & (1, 1, 1, 1, 1) & (0 | n_1 | n_2 | n_3 | n_4) \end{array}$$

We show specific examples with shifts  $k = 1, 3$  and  $5$  and signatures  $(5)$ ,  $(1, 1, 3)$  and  $(1, 1, 1, 1, 1)$ .

**Example 5.2.** We construct a  $(5, 1)$ -cyclic Maya diagram in the signature class  $(5)$  by choosing  $(n_1, n_2, n_3, n_4) = (2, 3, 1, 1)$ , which means that the first Maya diagram in the cycle is  $M_0 = \Xi(0, 2, 5, 6, 7)$ , depicted in the first row of Figure 5.1. The canonical flip sequence is  $\mu = (0, 2, 5, 6, 7)$ . We choose the permutation  $(34210)$ , which gives the chain of Maya diagrams shown in Figure 5.1. Note that the permutation specifies the sequence of block coordinates that get shifted by one at each step of the cycle.

This type of solutions with signature (5) were already studied in [29], and they are based on a genus 2 generalization of the generalized Hermite polynomials that appear in the solution of  $P_{IV}(A_2\text{-Painlevé})$ .

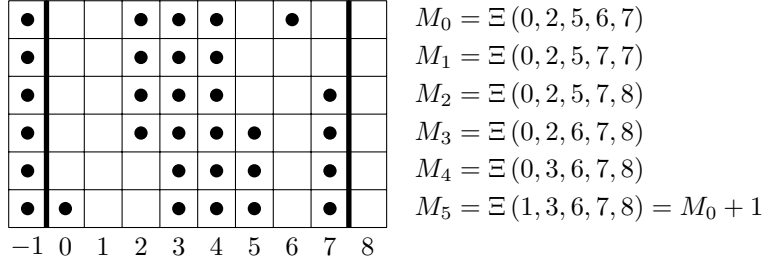


FIGURE 5.1. A Maya 5-cycle with shift  $k = 1$  for the choice  $(n_1, n_2, n_3, n_4) = (2, 3, 1, 1)$  and permutation  $\pi = (34210)$ .

We shall now provide the explicit construction of the rational solution to the  $A_4$ -Painlevé system (5.1), by using Proposition 3.16 and Proposition 2.3. The permutation  $\pi = (34210)$  on the canonical sequence  $\mu = (0, 2, 5, 6, 7)$  produces the flip sequence  $\mu_\pi = (6, 7, 5, 2, 0)$ , so that the values of the  $a_i$  parameters given by (3.18) become  $(a_0, a_1, a_2, a_3, a_4) = (-2, 4, 6, 4, -14)$ . The pseudo-Wronskians corresponding to each Maya diagram in the cycle are ordinary Wronskians, which will always be the case with the normalization imposed in Remark 4.15. They read (see Figure 5.1):

$$\begin{aligned}
 H_{M_0}(z) &= \text{Wr}(H_2, H_3, H_4, H_6), \\
 H_{M_1}(z) &= \text{Wr}(H_2, H_3, H_4), \\
 H_{M_2}(z) &= \text{Wr}(H_2, H_3, H_4, H_7), \\
 H_{M_3}(z) &= \text{Wr}(H_2, H_3, H_4, H_5, H_7), \\
 H_{M_4}(z) &= \text{Wr}(H_3, H_4, H_5, H_7),
 \end{aligned}$$

where  $H_n = H_n(z)$  is the  $n^{\text{th}}$  Hermite polynomial. Following Proposition 3.16, the rational solution to the dressing chain is given by the tuple  $(w_0, w_1, w_2, w_3, w_4 | a_0, a_1, a_2, a_3, a_4)$ , where  $a_i$  and  $w_i$  are given by (3.17)–(3.18) as:

$$\begin{aligned}
 w_0(z) &= z + \frac{d}{dz} \left[ \log H_{M_1}(z) - \log H_{M_0}(z) \right], & a_0 &= -2, \\
 w_1(z) &= -z + \frac{d}{dz} \left[ \log H_{M_2}(z) - \log H_{M_1}(z) \right], & a_1 &= 4, \\
 w_2(z) &= -z + \frac{d}{dz} \left[ \log H_{M_3}(z) - \log H_{M_2}(z) \right], & a_2 &= 6, \\
 w_3(z) &= z + \frac{d}{dz} \left[ \log H_{M_4}(z) - \log H_{M_3}(z) \right], & a_3 &= 4, \\
 w_4(z) &= -z + \frac{d}{dz} \left[ \log H_{M_0}(z) - \log H_{M_4}(z) \right], & a_4 &= -14.
 \end{aligned}$$

Finally, Proposition 2.3 implies that the corresponding rational solution to the  $A_4$ -Painlevé system (1.7) is given by the tuple  $(f_0, f_1, f_2, f_3, f_4 | \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ , where

$$\begin{aligned} f_0(z) &= \frac{d}{dz} \left[ \log H_{M_2}(c_1 z) - \log H_{M_0}(c_1 z) \right], & \alpha_0 &= 1, \\ f_1(z) &= z + \frac{d}{dz} \left[ \log H_{M_3}(c_1 z) - \log H_{M_1}(c_1 z) \right], & \alpha_1 &= -2, \\ f_2(z) &= \frac{d}{dz} \left[ \log H_{M_4}(c_1 z) - \log H_{M_2}(c_1 z) \right], & \alpha_2 &= -3, \\ f_3(z) &= \frac{d}{dz} \left[ \log H_{M_0}(c_1 z) - \log H_{M_3}(c_1 z) \right], & \alpha_3 &= -2, \\ f_4(z) &= \frac{d}{dz} \left[ \log H_{M_1}(c_1 z) - \log H_{M_4}(c_1 z) \right], & \alpha_4 &= 7. \end{aligned}$$

with  $c_1^2 = -\frac{1}{2}$ .

**Example 5.3.** We construct a degenerate example belonging to the (5) signature class, by choosing  $(n_1, n_2, n_3, n_4) = (1, 1, 2, 0)$ . The presence of  $n_4 = 0$  means that the first Maya diagram has genus 1 instead of the generic genus 2, with block coordinates given by  $M_0 = \Xi(0, 1, 2, 4, 4)$ . The canonical flip sequence  $\mu = (0, 1, 2, 4, 4)$  contains two flips at the same site, so it is not unique. Choosing the permutation (42130) produces the chain of Maya diagrams shown in Figure 5.2. The explicit construction of the rational solutions follows the same steps as in the previous example, and we shall omit it here. It is worth noting, however, that due to the degenerate character of the chain, three linear combinations of  $f_0, \dots, f_4$  will provide a solution to the lower rank  $A_2$ -Painlevé. If the two flips at the same site are performed consecutively in the cycle, the embedding of  $A_2^{(1)}$  into  $A_4^{(1)}$  is trivial and corresponds to setting two consecutive  $f_i$  to zero. This is not the case in this example, as the flip sequence is  $\mu_\pi = (4, 2, 1, 4, 0)$ , which produces a non-trivial embedding.

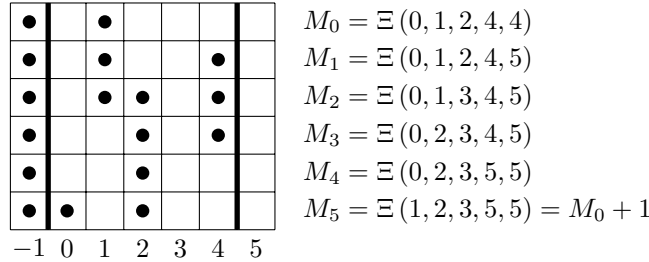


FIGURE 5.2. A degenerate Maya 5-cycle with  $k = 1$  for the choice  $(n_1, n_2, n_3, n_4) = (1, 1, 2, 0)$  and permutation  $\pi = (42130)$ .

**Example 5.4.** We construct a (5, 3)-cyclic Maya diagram in the signature class (1, 1, 3) by choosing  $(n_1, n_2, n_3, n_4) = (3, 1, 1, 2)$ , which means that the first Maya diagram has 3-block coordinates  $(0|3|1, 2, 4)$ . The canonical flip sequence is given by  $\mu = \Theta(0|3|1, 2, 4) = (0, 10, 5, 8, 14)$ . The permutation (41230) gives the chain of Maya diagrams shown in Figure 5.3. Note that, as in Example 5.2, the permutation specifies the order in which the 3-block coordinates are shifted by +1 in the subsequent steps of the cycle. This type of solutions in the signature class (1, 1, 3) were not given in [29], and they are new to the best of our knowledge.

We proceed to build the explicit rational solution to the  $A_4$ -Painlevé system (5.1). In this case, the permutation  $\pi = (41230)$  on the canonical sequence  $\mu = (0, 10, 5, 8, 14)$  produces the flip sequence  $\mu_\pi = (14, 10, 5, 8, 0)$ , so that the values of the  $a_i$  parameters given by (3.18) become  $(a_0, a_1, a_2, a_3, a_4) = (8, 10, -6, 16, -34)$ . The pseudo-Wronskians corresponding to each Maya diagram in the cycle are



**Example 5.5.** We construct a  $(5, 5)$ -cyclic Maya diagram in the signature class  $(1, 1, 1, 1, 1)$  by choosing  $(n_1, n_2, n_3, n_4) = (2, 3, 0, 1)$ , which means that the first Maya diagram has 5-block coordinates  $(0|2|3|0|1)$ . The canonical flip sequence is given by  $\boldsymbol{\mu} = \Theta(0|2|3|0|1) = (0, 11, 17, 3, 9)$ . The permutation  $(32410)$  gives the chain of Maya diagrams shown in Figure 5.4. Note that, as it happens in the previous examples, the permutation specifies the order in which the 5-block coordinates are shifted by  $+1$  in the subsequent steps of the cycle. This type of solutions with signature  $(1, 1, 1, 1, 1)$  were already studied in [29], and they are based on a generalization of the Okamoto polynomials that appear in the solution of  $P_{\text{IV}}(A_2\text{-Painlevé})$ .

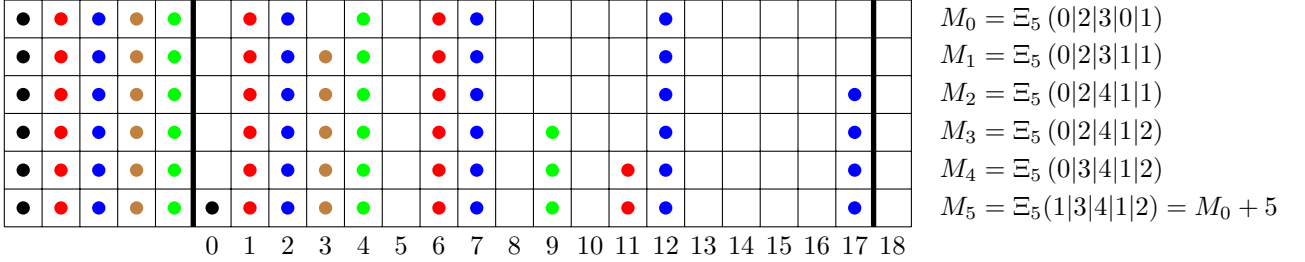


FIGURE 5.4. A Maya 5-cycle with shift  $k = 5$  for the choice  $(n_1, n_2, n_3, n_4) = (2, 3, 0, 1)$  and permutation  $\boldsymbol{\pi} = (32410)$ .

We proceed to build the explicit rational solution to the  $A_4$ -Painlevé system (5.1). In this case, the permutation  $\boldsymbol{\pi} = (32410)$  on the canonical sequence  $\boldsymbol{\mu} = (0, 11, 17, 3, 9)$  produces the flip sequence  $\boldsymbol{\mu}_{\boldsymbol{\pi}} = (3, 17, 9, 11, 0)$ , so that the values of the  $a_i$  parameters given by (3.18) become  $(a_0, a_1, a_2, a_3, a_4) = (-28, 16, -4, 22, -16)$ . The pseudo-Wronskians corresponding to each Maya diagram in the cycle are ordinary Wronskians, which will always be the case with the normalization imposed in Remark 4.15. They read:

$$\begin{aligned} H_{M_0}(z) &= \text{Wr}(H_1, H_2, H_4, H_6, H_7, H_{12}), \\ H_{M_1}(z) &= \text{Wr}(H_1, H_2, H_3, H_4, H_6, H_7, H_{12}), \\ H_{M_2}(z) &= \text{Wr}(H_1, H_2, H_3, H_4, H_6, H_7, H_{12}, H_{17}), \\ H_{M_3}(z) &= \text{Wr}(H_1, H_2, H_3, H_4, H_6, H_7, H_9, H_{12}, H_{17}), \\ H_{M_4}(z) &= \text{Wr}(H_1, H_2, H_3, H_4, H_6, H_7, H_9, H_{11}, H_{12}, H_{17}), \end{aligned}$$

where  $H_n = H_n(z)$  is the  $n$ -th Hermite polynomial. The rational solution to the dressing chain is given by the tuple  $(w_0, w_1, w_2, w_3, w_4|a_0, a_1, a_2, a_3, a_4)$ , where  $a_i$  and  $w_i$  are given by (3.17)–(3.18) as:

$$\begin{aligned} w_0(z) &= -z + \frac{d}{dz} \left[ \log H_{M_1}(z) - \log H_{M_0}(z) \right], & a_0 &= -28 \\ w_1(z) &= -z + \frac{d}{dz} \left[ \log H_{M_2}(z) - \log H_{M_1}(z) \right], & a_1 &= 16, \\ w_2(z) &= -z + \frac{d}{dz} \left[ \log H_{M_3}(z) - \log H_{M_2}(z) \right], & a_2 &= -4, \\ w_3(z) &= -z + \frac{d}{dz} \left[ \log H_{M_4}(z) - \log H_{M_3}(z) \right], & a_3 &= 22 \\ w_4(z) &= -z + \frac{d}{dz} \left[ \log H_{M_0}(z) - \log H_{M_4}(z) \right], & a_4 &= -16. \end{aligned}$$

Finally, Proposition 3.16 implies that the corresponding rational solution to the  $A_4$ -Painlevé system (1.7) is given by the tuple  $(f_0, f_1, f_2, f_3, f_4 | \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ , where

$$\begin{aligned} f_0(z) &= \frac{1}{5}z + \frac{d}{dz} \left[ \log H_{M_2}(c_3 z) - \log H_{M_0}(c_3 z) \right], & \alpha_0 &= \frac{14}{5}, \\ f_1(z) &= \frac{1}{5}z + \frac{d}{dz} \left[ \log H_{M_3}(c_3 z) - \log H_{M_1}(c_3 z) \right], & \alpha_1 &= -\frac{8}{5}, \\ f_2(z) &= \frac{1}{5}z + \frac{d}{dz} \left[ \log H_{M_4}(c_3 z) - \log H_{M_2}(c_3 z) \right], & \alpha_2 &= \frac{2}{5}, \\ f_3(z) &= \frac{1}{5}z + \frac{d}{dz} \left[ \log H_{M_0}(c_3 z) - \log H_{M_3}(c_3 z) \right], & \alpha_3 &= -\frac{11}{5}, \\ f_4(z) &= \frac{1}{5}z + \frac{d}{dz} \left[ \log H_{M_1}(c_3 z) - \log H_{M_4}(c_3 z) \right], & \alpha_4 &= \frac{8}{5}. \end{aligned}$$

with  $c_3^2 = -\frac{1}{10}$ .

**5.1. Zeros of the special polynomials in the  $A_4$  rational solutions.** The zeros of Okamoto and generalized Hermite polynomials that appear in the rational solutions to  $P_{IV}$  are known to form very regular patterns in the complex plane, [21]. In this section we show the equivalent patterns for their  $A_4$  counterparts, which are also very regular but show a richer structure.

Following the notation above, we label a Maya diagram in standard form by specifying its signature as a superscript, and 4 non-negative integers  $(n_1, n_2, n_3, n_4)$  that determine the  $k$ -block coordinates as specified by Theorem 5.1. More specifically, we can write the sequence of positive integers that belong to  $M$  in the following manner:

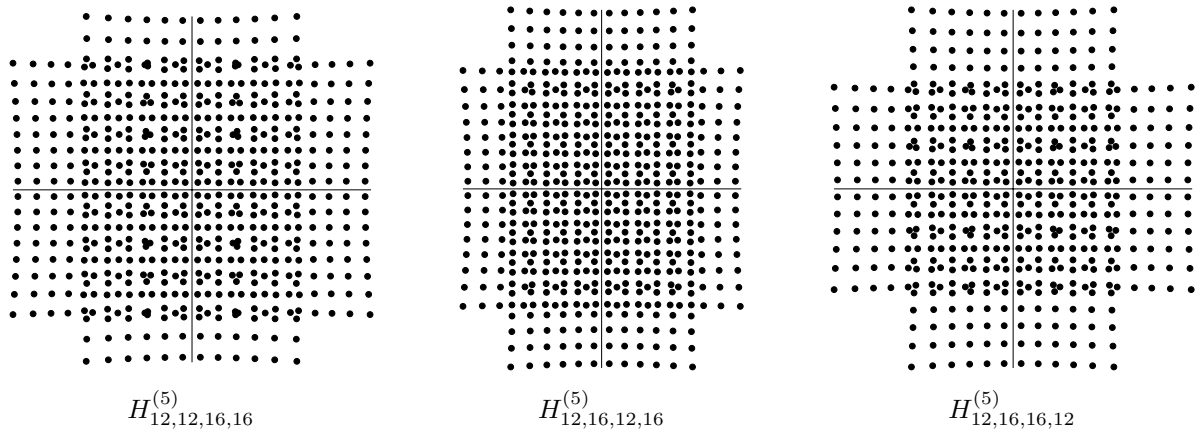
$$\begin{aligned} M_+^{(5)}(n_1, n_2, n_3, n_4) &= \left\{ n_1 + j \right\}_{j=0}^{n_2-1} \cup \left\{ n_1 + n_2 + n_3 + j \right\}_{j=0}^{n_4-1} \\ M_+^{(3,1,1)}(n_1, n_2, n_3, n_4) &= \left\{ 3(n_1 + j) \right\}_{j=0}^{n_2-1} \cup \left\{ 1 + 3j \right\}_{j=0}^{n_3-1} \cup \left\{ 2 + 3j \right\}_{j=0}^{n_4-1} \\ M_+^{(1,1,1,1)}(n_1, n_2, n_3, n_4) &= \left\{ 1 + 5j \right\}_{j=0}^{n_1-1} \cup \left\{ 2 + 5j \right\}_{j=0}^{n_2-1} \cup \left\{ 3 + 5j \right\}_{j=0}^{n_3-1} \cup \left\{ 4 + 5j \right\}_{j=0}^{n_4-1} \end{aligned}$$

Likewise, we denote by  $H^{(s)}(n_1, n_2, n_3, n_4)$  the corresponding Hermite Wronskians for each signature  $s$ , i.e.

$$\begin{aligned} H_{n_1, n_2, n_3, n_4}^{(5)}(z) &= \text{Wr} \left[ \left\{ H_{n_1+j}(z) \right\}_{j=0}^{n_2-1}, \left\{ H_{n_1+n_2+n_3+j}(z) \right\}_{j=0}^{n_4-1} \right], \\ H_{n_1, n_2, n_3, n_4}^{(3,1,1)}(z) &= \text{Wr} \left[ \left\{ H_{3(n_1+j)}(z) \right\}_{j=0}^{n_2-1}, \left\{ H_{1+3j}(z) \right\}_{j=0}^{n_3-1}, \left\{ H_{2+3j}(z) \right\}_{j=0}^{n_4-1} \right], \\ H_{n_1, n_2, n_3, n_4}^{(1,1,1,1)}(z) &= \text{Wr} \left[ \left\{ H_{1+5j}(z) \right\}_{j=0}^{n_1-1}, \left\{ H_{2+5j}(z) \right\}_{j=0}^{n_2-1}, \left\{ H_{3+5j}(z) \right\}_{j=0}^{n_3-1}, \left\{ H_{4+5j}(z) \right\}_{j=0}^{n_4-1} \right]. \end{aligned}$$

From the observation of these plots it is clear that the geometric distribution of the zeros on the complex plane follows some regular patterns that call for an explanation. In some cases, specially in Figure 5.5 we observe two overlapping patterns that seems to suggest an approximate factorization of  $H_{n_1, n_2, n_3, n_4}^{(5)}(z)$  into the product of two generalized Hermite Wronskians  $H_{m_1, m_2}^{(3)}(z)$  and  $H_{m_3, m_4}^{(3)}(z)$ . More generally, the approximate correspondence between the Young diagrams of the partitions that determine the sequence of Hermite polynomials in the Wronskian, and the position of the zeros in the complex plane was observed in [28]. A detailed study of the zeros of these Hermite Wronskians is currently under investigation and we shall not pursue it further here.

We stress that all of these zeros are conjectured by Veselov to be simple [28], except the zero at the origin whose multiplicity is a triangular number, [12]. The number of zeros in the real line, and

FIGURE 5.5. Zeros of Hermite Wronskians  $H_{n_1, n_2, n_3, n_4}^{(5)}(z)$  for different values of  $(n_1, n_2, n_3, n_4)$ .

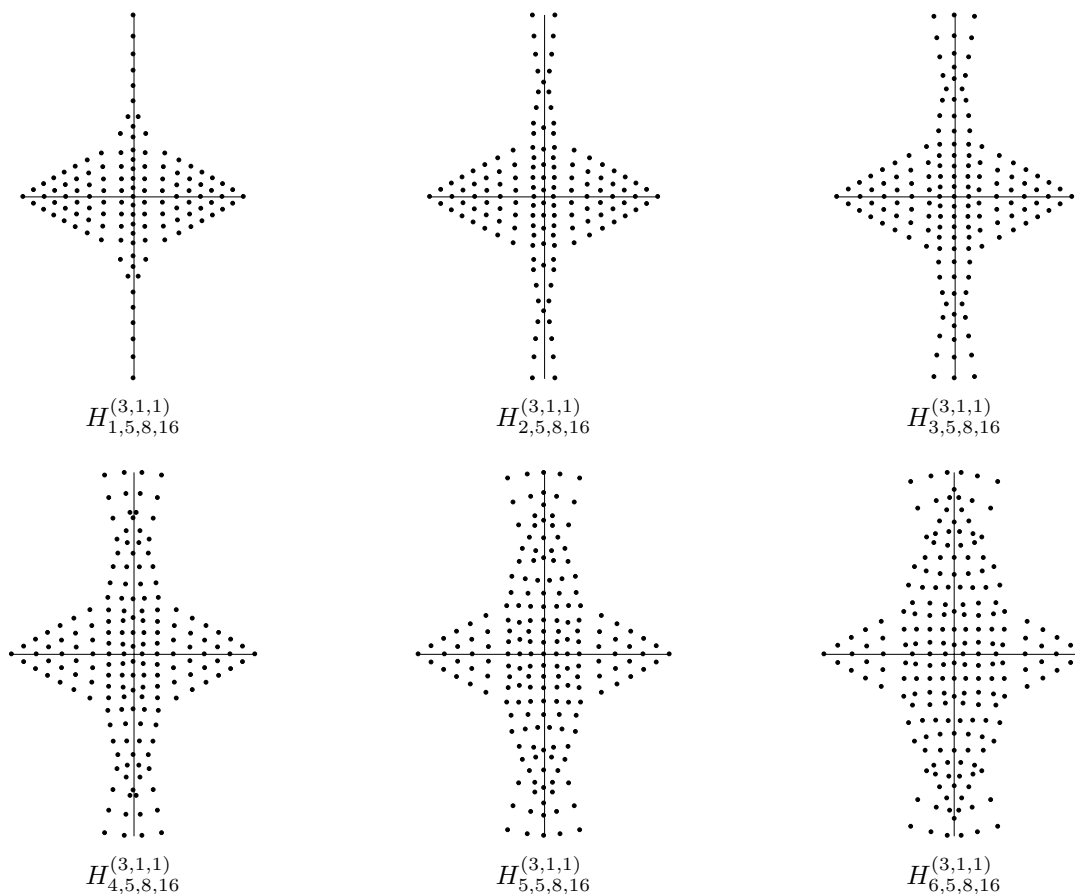
thus the real poles of the rational solutions can be calculated as a function of  $(n_1, n_2, n_3, n_4)$  for each signature class by applying the formulas derived in [33].

## 6. ACKNOWLEDGEMENTS

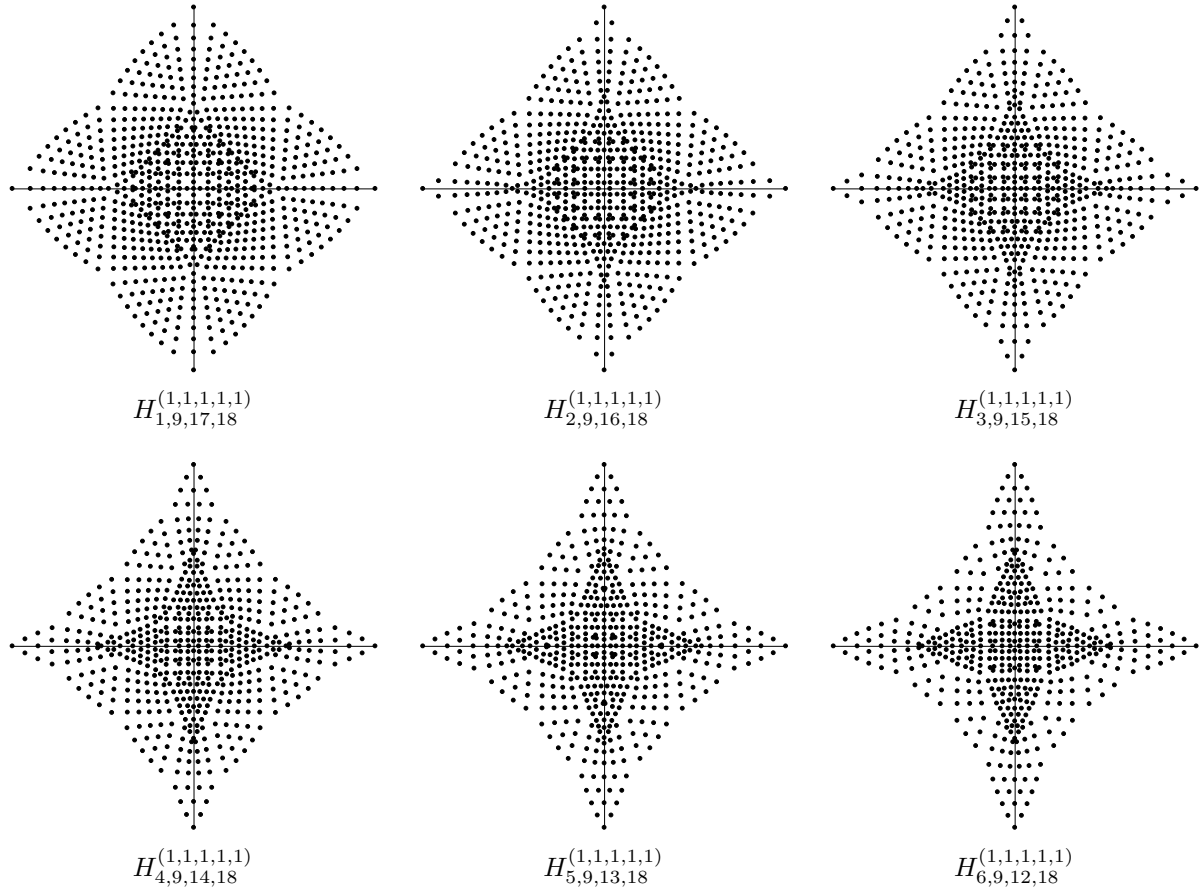
The research of DGU has been supported in part by Spanish MINECO-FEDER Grants MTM2015-65888-C4-3 and PGC2018-096504-B-C33, and by the ICMAT-Severo Ochoa project SEV-2015-0554. The research of RM was supported in part by NSERC grant RGPIN-228057-2009. DGU would like to thank the University of Kent for their hospitality during his research stay in Spring 2017 where part of these results were obtained, and the Spanish MINECO program Salvador de Madariaga that provided the necessary financial support. Interesting discussions with Andy Hone, Claire Dunning, Kerstin Jordaan and Galina Filipuk are also gratefully acknowledged. Galina pointed us to the fact that a Wronskian representation for the  $k = 3$  shift rational solutions in the  $A_4$ -Painlevé case was not known, and this observation triggered a good amount of the research done in this paper. PAC would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme “Complex analysis: techniques, applications and computations” when some of the work on this paper was undertaken. This work was supported by EPSRC grant number EP/R014604/1.

## REFERENCES

- [1] Mark Adler and Jürgen Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, Communications in Mathematical Physics **61** (1978), no. 1, 1–30.
- [2] V. È. Adler, *A modification of Crum’s method*, Theoret. and Math. Phys. **101** (1994), no. 3, 1381–1386.
- [3] ———, *Nonlinear chains and Painlevé equations*, Phys. D **73** (1994), no. 4, 335–351.
- [4] George E. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998. MR 1634067
- [5] George E. Andrews and Kimmo Eriksson, *Integer partitions*, Cambridge University Press, Cambridge, 2004. MR 2122332
- [6] B. Bagchi, Y. Grandati, and C. Quesne, *Rational extensions of the trigonometric Darboux-Pöschl-Teller potential based on para-Jacobi polynomials*, J. Math. Phys. **56** (2015), no. 6, 062103.
- [7] I. V. Barashenkov and D. E. Pelinovsky, *Exact vortex solutions of the complex sine-Gordon theory on the plane*, Phys. Lett. B **436** (1998), no. 1-2, 117–124. MR 1649103
- [8] L. Bass, J. J. C. Nimmo, C. Rogers, and W. K. Schief, *Electrical structures of interfaces: a Painlevé II model*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **466** (2010), no. 2119, 2117–2136. MR 2652736
- [9] D Bermúdez and Fernández D J, *Complex solutions to the Painlevé IV equation through supersymmetric quantum mechanics*, AIP Conference Proceedings, vol. 1420, AIP, 2012, pp. 47–51.

FIGURE 5.6. Zeros of Hermite Wronskians  $H_{n_1, n_2, n_3, n_4}^{(3,1,1)}(z)$  for different values of  $(n_1, n_2, n_3, n_4)$ .

- [10] David Bermúdez, *Complex SUSY transformations and the Painlevé IV equation*, SIGMA **8** (2012), 069.
- [11] Marco Bertola and Thomas Bothner, *Zeros of large degree Vorob'ev-Yablonski polynomials via a Hankel determinant identity*, Int. Math. Res. Not. IMRN (2015), no. 19, 9330–9399. MR 3431594
- [12] Niels Bonneux, Clare Dunning, and Marco Stevens, *Coefficients of Wronskian Hermite polynomials*, arXiv preprint arXiv:1909.03874 (2019).
- [13] Robert J. Buckingham, *Large-degree asymptotics of rational Painlevé-IV functions associated to generalized Hermite polynomials*, Int. Math. Res. Not. IMRN, DOI: 10.1093/imrn/rny172.
- [14] Robert J. Buckingham and Peter D. Miller, *The sine-Gordon equation in the semiclassical limit: critical behavior near a separatrix*, J. Anal. Math. **118** (2012), no. 2, 397–492. MR 3000688
- [15] ———, *Large-degree asymptotics of rational Painlevé-II functions: noncritical behaviour*, Nonlinearity **27** (2014), no. 10, 2489–2578. MR 3265723
- [16] ———, *Large-degree asymptotics of rational Painlevé-II functions: critical behaviour*, Nonlinearity **28** (2015), no. 6, 1539–1596. MR 3350600
- [17] JL Burchnall and TW Chaundy, *A set of differential equations which can be solved by polynomials*, Proceedings of the London Mathematical Society **2** (1930), no. 1, 401–414.
- [18] Hongmei Chen, Min Chen, Gordon Blower, and Yang Chen, *Single-user MIMO system, Painlevé transcendents, and double scaling*, J. Math. Phys. **58** (2017), no. 12, 123502, 24. MR 3734945
- [19] Yang Chen and M. V. Feigin, *Painlevé IV and degenerate Gaussian unitary ensembles*, J. Phys. A **39** (2006), no. 40, 12381–12393.

FIGURE 5.7. Zeros of Hermite Wronskians  $H_{n_1, n_2, n_3, n_4}^{(1,1,1,1)}(z)$  for different values of  $(n_1, n_2, n_3, n_4)$ .

- [20] Peter A. Clarkson, *Painlevé equations — nonlinear special functions*, J. Comput. Appl. Math. **153** (2003), no. 1-2, 127–140.
- [21] ———, *The fourth Painlevé equation and associated special polynomials*, J. Math. Phys. **44** (2003), no. 11, 5350–5374.
- [22] ———, *Special polynomials associated with rational solutions of the defocusing nonlinear Schrödinger equation and the fourth Painlevé equation*, European J. Appl. Math. **17** (2006), no. 3, 293–322.
- [23] ———, *Special polynomials associated with rational solutions of the Painlevé equations and applications to soliton equations*, Comput. Methods Funct. Theory **6** (2006), no. 2, 329–401. MR 2291140
- [24] ———, *Vortices and polynomials*, Stud. Appl. Math. **123** (2009), no. 1, 37–62. MR 2538285
- [25] Peter A. Clarkson and Kerstin Jordaan, *The relationship between semiclassical Laguerre polynomials and the fourth Painlevé equation*, Constr. Approx. **39** (2014), no. 1, 223–254.
- [26] Peter A. Clarkson and Elizabeth L. Mansfield, *The second Painlevé equation, its hierarchy and associated special polynomials*, Nonlinearity **16** (2003), no. 3, R1–R26. MR 1975781
- [27] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Comm. Math. Phys. **103** (1986), no. 2, 177–240. MR 826863
- [28] G. Felder, A. D. Hemery, and A. P. Veselov, *Zeros of Wronskians of Hermite polynomials and Young diagrams*, Phys. D **241** (2012), no. 23-24, 2131–2137. MR 2998116
- [29] Galina Filipuk and Peter A. Clarkson, *The symmetric fourth Painlevé hierarchy and associated special polynomials*, Stud. Appl. Math. **121** (2008), no. 2, 157–188.

- [30] Athanassios S. Fokas, Alexander R. Its, Andrei A. Kapaev, and Victor Yu. Novokshenov, *Painlevé transcendents. The Riemann-Hilbert approach*, Mathematical Surveys and Monographs, vol. 128, American Mathematical Society, Providence, RI, 2006. MR 2264522
- [31] P. J. Forrester and N. S. Witte, *Application of the  $\tau$ -function theory of Painlevé equations to random matrices: PIV, PII and the GUE*, Comm. Math. Phys. **219** (2001), no. 2, 357–398. MR 1833807
- [32] Ma Ángeles García-Ferrero, David Gómez-Ullate, and Robert Milson, *A bochner type characterization theorem for exceptional orthogonal polynomials*, Journal of Mathematical Analysis and Applications **472** (2019), no. 1, 584–626.
- [33] Ma Ángeles García-Ferrero and David Gómez-Ullate, *Oscillation theorems for the Wronskian of an arbitrary sequence of eigenfunctions of Schrödinger’s equation*, Lett. Math. Phys. **105** (2015), no. 4, 551–573.
- [34] David Gómez-Ullate, Yves Grandati, and Robert Milson, *Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials*, J. Phys. A **47** (2013), no. 1, 015203.
- [35] ———, *Durfee rectangles and pseudo-Wronskian equivalences for Hermite polynomials*, Stud. Appl. Math. **141** (2018), no. 4, 596–625.
- [36] David Gómez-Ullate, Niky Kamran, and Robert Milson, *Supersymmetry and algebraic Darboux transformations*, J. Phys. A **37** (2004), no. 43, 10065.
- [37] ———, *The Darboux transformation and algebraic deformations of shape-invariant potentials*, J. Phys. A **37** (2004), no. 5, 1789.
- [38] ———, *An extended class of orthogonal polynomials defined by a Sturm-Liouville problem*, J. Math. Anal. Appl. **359** (2009), no. 1, 352–367.
- [39] ———, *An extension of Bochner’s problem: exceptional invariant subspaces*, J. Approx. Theory **162** (2010), no. 5, 987–1006.
- [40] ———, *A conjecture on exceptional orthogonal polynomials*, Found. Comput. Math. **13** (2013), no. 4, 615–666.
- [41] Yves Grandati, *Solvable rational extensions of the isotonic oscillator*, Ann. Physics **326** (2011), no. 8, 2074–2090.
- [42] ———, *Multistep DBT and regular rational extensions of the isotonic oscillator*, Ann. Physics **327** (2012), no. 10, 2411–2431.
- [43] Valerii I Gromak, Ilpo Laine, and Shun Shimomura, *Painlevé differential equations in the complex plane*, vol. 28, Walter de Gruyter, 2008.
- [44] Kenji Kajiwara and Yasuhiro Ohta, *Determinant structure of the rational solutions for the Painlevé II equation*, J. Math. Phys. **37** (1996), no. 9, 4693–4704.
- [45] ———, *Determinant structure of the rational solutions for the Painlevé IV equation*, J. Phys. A **31** (1998), no. 10, 2431.
- [46] M. G. Krein, *On a continual analogue of a Christoffel formula from the theory of orthogonal polynomials*, Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 970–973. MR 0091396
- [47] Ian Marquette and Christiane Quesne, *New ladder operators for a rational extension of the harmonic oscillator and superintegrability of some two-dimensional systems*, J. Math. Phys. **54** (2013), no. 10, 102102, 12. MR 3134580
- [48] ———, *Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators*, J. Phys. A **46** (2013), no. 15, 155201.
- [49] ———, *Connection between quantum systems involving the fourth Painlevé transcendent and  $k$ -step rational extensions of the harmonic oscillator related to Hermite exceptional orthogonal polynomial*, J. Math. Phys. **57** (2016), no. 5, 052101, 15. MR 3501792
- [50] Davide Masoero and Pieter Roffelsen, *Poles of Painlevé IV rationals and their distribution*, SIGMA **14** (2018), Paper No. 002, 49. MR 3742702
- [51] ———, *Roots of generalised Hermite polynomials when both parameters are large*, arXiv preprint arXiv:1907.08552 (2019).
- [52] J. Mateo and J. Negro, *Third-order differential ladder operators and supersymmetric quantum mechanics*, J. Phys. A **41** (2008), no. 4, 045204, 28. MR 2451071
- [53] Kazuhide Matsuda, *Rational solutions of the Noumi and Yamada system of type  $A_4^{(1)}$* , J. Math. Phys. **53** (2012), no. 2, 023504.
- [54] Masatoshi Noumi, *Painlevé equations through symmetry*, vol. 223, Springer Science & Business, 2004.
- [55] Masatoshi Noumi and Yasuhiko Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*, Nagoya Math. J. **153** (1999), 53–86.
- [56] V. Yu. Novokshenov and A. A. Shchelkonogov, *Double scaling limit in the Painlevé IV equation and asymptotics of the Okamoto polynomials*, Spectral theory and differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 233, Amer. Math. Soc., Providence, RI, 2014, pp. 199–210. MR 3307781
- [57] Victor Yu. Novokshenov, *Generalized Hermite polynomials and monodromy-free Schrödinger operators*, SIGMA **14** (2018), 106, 13 pages. MR 3859422
- [58] A. A. Oblomkov, *Monodromy-free Schrödinger operators with quadratically increasing potentials*, Theoret. and Math. Phys. **121** (1999), no. 3, 1574–1584.

- [59] Satoru Odake and Ryu Sasaki, *Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials*, Phys. Lett. B **702** (2011), no. 2-3, 164–170.
- [60] ———, *Extensions of solvable potentials with finitely many discrete eigenstates*, J. Phys. A **46** (2013), no. 23, 235205.
- [61] ———, *Krein–Adler transformations for shape-invariant potentials and pseudo virtual states*, J. Phys. A **46** (2013), no. 24, 245201.
- [62] Kazuo Okamoto, *Studies on the Painlevé equations. III. Second and fourth Painlevé equations,  $P_{II}$  and  $P_{IV}$* , Math. Ann. **275** (1986), no. 2, 221–255. MR 854008
- [63] Jørn Børling Olsson, *Combinatorics and representations of finite groups*, Fachbereich Mathematik [Lecture Notes in Mathematics], vol. 20, Universität Essen, 1994.
- [64] N. Olver and I. V. Barashenkov, *The complex sine-Gordon-2 equation: a new algorithm for obtaining multivortex solution on the plane*, Theoret. and Math. Phys. **144** (2005), no. 2, 1223–1226.
- [65] Colin Rogers, *Moving boundary problems for the Harry Dym equation and its reciprocal associates*, Z. Angew. Math. Phys. **66** (2015), no. 6, 3205–3220. MR 3428461
- [66] ———, *On a class of moving boundary problems for the potential mkdV equation: conjugation of Bäcklund and reciprocal transformations*, Ric. Mat. **65** (2016), no. 2, 563–577. MR 3572529
- [67] ———, *Moving boundary problems for an extended Dym equation. Reciprocal connection*, Meccanica **52** (2017), no. 15, 3531–3540. MR 3719648
- [68] Colin Rogers and Peter A. Clarkson, *Ermakov-Painlevé II symmetry reduction of a Korteweg capillarity system*, SIGMA **13** (2017), Paper No. 018, 20. MR 3626297
- [69] ———, *Ermakov-Painlevé II reduction in cold plasma physics. Application of a Bäcklund transformation*, J. Non-linear Math. Phys. **25** (2018), no. 2, 247–261. MR 3776560
- [70] Amit Sen, Andrew N. W. Hone, and Peter A. Clarkson, *Darboux transformations and the symmetric fourth Painlevé equation*, J. Phys. A **38** (2005), no. 45, 9751–9764.
- [71] Kanehisa Takasaki, *Spectral curve, Darboux coordinates and Hamiltonian structure of periodic dressing chains*, Comm. Math. Phys. **241** (2003), no. 1, 111–142.
- [72] Teruhisa Tsuda, *Universal characters, integrable chains and the Painlevé equations*, Adv. Math. **197** (2005), no. 2, 587–606.
- [73] Hiroshi Umemura, *Painlevé equations in the past 100 years*, A.M.S. Translations **204** (2001), 81–110.
- [74] Walter Van Assche, *Orthogonal polynomials and Painlevé equations*, Australian Mathematical Society Lecture Series, vol. 27, Cambridge University Press, Cambridge, 2018. MR 3729446
- [75] A. P. Veselov and A. B. Shabat, *Dressing chains and the spectral theory of the Schrödinger operator*, Funct. Anal. Appl. **27** (1993), no. 2, 81–96.
- [76] A. P. Vorob’ev, *On the rational solutions of the second Painlevé equation*, Diff. Eqns. **1** (1965), no. 1, 58–59.
- [77] Ralph Willox and Jarmo Hietarinta, *Painlevé equations from Darboux chains. I.  $P_{III}$ – $P_V$* , J. Phys. A **36** (2003), no. 42, 10615–10635. MR 2024916
- [78] A. I. Yablonskii, *On rational solutions of the second Painlevé equation*, Vesti Akad. Navuk. BSSR Ser. Fiz. Tkh. Nauk. **3** (1959), 30–35.

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