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Coulomb’s theory of arches in Spain ca. 1800: the manuscript of Joaquín Monasterio

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ABSTRACT: The Spanish engineer Joaquín Monasterio wrote a “New theory on the thrust of vaults” between 1805-1810. It was the first application of Coulomb's theory of vaults, which he outlined in 1773. The memoir by Monasterio is remarkable also for its generality and originality. He was, for example the first to calculate correctly the mathematical limit thickness of a semicircular arch and to tackle the problem of the limit analysis of domes. The memoire has remained unpublished and unknown until recently. The purpose of this paper is to call the attention to a contribution which merits an honourable place in the history of vault theory.

1 JOAQUÍN MONASTERIO

The unpublished manuscript entitled *Nueva Teórica sobre el Empuje de Bóvedas* is in the Library of the Escuela de Ingenieros de Caminos, Canales y Puertos of the Universidad Politécnica de Madrid, and was rediscovered by one of the authors in 1993 (Huerta and Foce, 2003). The name *Monasterio* appears on the front page of the document, referring, it seems evident, to the author’s name. In the opinion of Fernando Sáenz Ridruejo, this name corresponds, most probably, to Civil Engineer Joaquín Monasterio. This obscure figure is known to have been part of the first class that graduated from the ‘Antigua’ Escuela de Caminos, established by Agustín de Betancourt in 1802. The information about Monasterio is scarce. Rumeu simple mentions him, and we are indebted to F. Sáenz for the following information: Graduated in 1804; he features in a proposal for the Academia de la Ciencia in 1809; and he was promoted to ‘Ingeniero de 1ª Clase’ in 1810. There is no further information about him after this date.

The Escuela de Caminos was closed from 1805 to 1806. Sáenz proposes that Monasterio could have written his *Teorica* during this time. This date agrees with the evidence contained within the manuscript: Monasterio cites various works on the subject of the statics of arches. Among these references, he includes Coulomb and Rondelet (first edition of the third vol. on vaults 1805), but doesn’t mention Gauthey, a treatise which was inmediately accepted as the best reference in bridge design. It appears, then that the manuscript could be dated between 1805-1810.

There is no doubt that the manuscript arrived in the hands of Eduardo Saavedra, one of the most prominent Spanish engineers of the 19th century. In the first numbered page of the manuscript (the manuscript was unnumbered, and page count is taken from the page immediately after the tittle page), the following note by Saavedra can be read: ‘*This work is by professor Monasterio of the early Escuela de Caminos, and it was given to me as a present by my master D. Francisco de Travesedo.*’. Saavedra includes the manuscript in the list of Spanish bibliography on the subject of vaults he writes for the Clairac Dictionary). The manuscript belonged to Saavedra’s private library, and was later donated (unknown date) to the Library of the Escuela de Caminos.

2 THE MANUSCRIPT

The *Nueva Teórica sobre el Empuje de Bóvedas* comprises an introduction, followed by three chapters devoted to the collapse analysis of arches and a fourth chapter on the collapse mechanisms of abutments. It is a handwritten document, with long formulations and very few corrections, including two plates (see Fig.1) with a total of 25 figures.

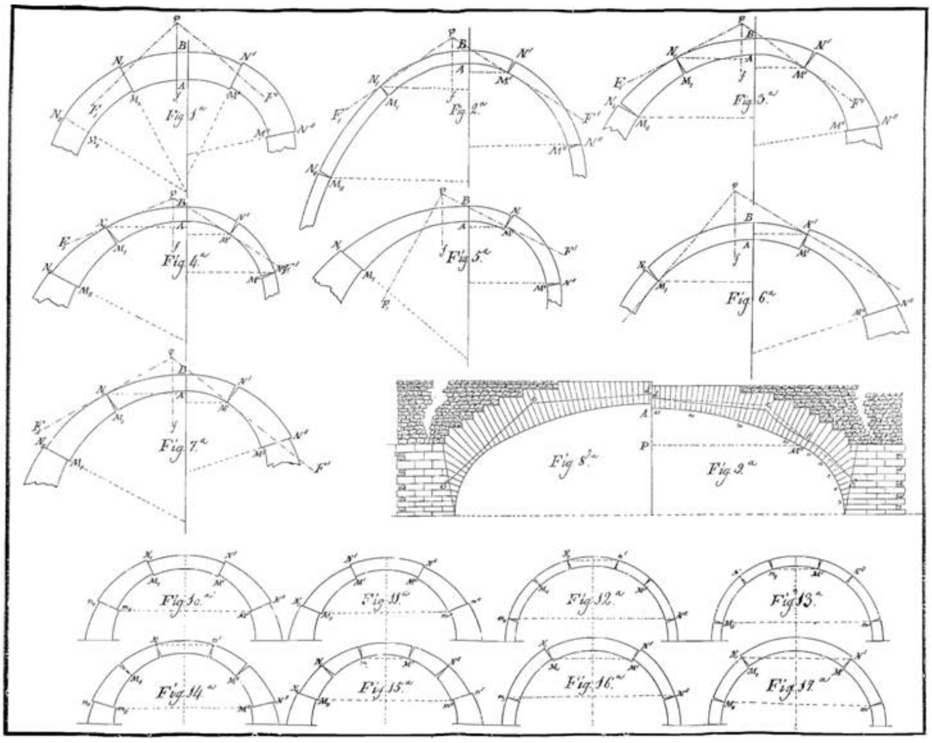

Figure 1 : Monasterio's Manuscript Plate I, showing the 7 possible collapse mechanisms of arches

Monasterio repeatedly applies the same methodology in his analyses throughout the text: he devises a collapse mechanism for the structure under consideration (vaults or piers) and applies equilibrium at the point of collapse to obtain the limit values of the design parameters he is trying to determine. He adopts the maxima and minima approach defined by Coulomb, of whom he says:

'…it is now easy to generate a fair opinion of the merit of the best works written on the thrust of vaults by referring to the procedure the authors should have follow in order that their calculations yielded useful practical applications. We can say that M. Coulomb would have been the best in developing the research…' (Monasterio, 2010)

In contrast with precedent investigations on the mechanics of vaults, Monasterio analyses generic non-symmetric arches, of non-uniform thickness. According to Huerta and Foce, he is probably the first to tackle the problem: His aim is to generate general expressions for preventing all possible collapse mechanisms, applicable to any arch, of any shape and thickness. He only establishes one condition the overall shape of the arch must meet: all the voussoirs the arch can be divided into must be so that the vertical line through their centre of gravity must fall between their two intrados corners, or between the lower intrados and extrados corners, but never fall beyond the lower extrados corner.

2.1 Failure mechanisms

In the *Introducción* to his manuscript, Monasterio clearly defines his approach to the problem of the mechanics of vaults:

'Therefore, if the objective of the theory of vaults is to prevent arches from collapsing or suffering alterations that could potentially diminish their strength, it is clear that this objective won't be properly satisfied if all the possible movements of the different parts of the arch are not first located. The conditions for preventing each of the possible movements will be found next' (Monasterio, 2010).

Vaults fail by ‘moving’, that is, by becoming mechanisms that can move freely, and, hence, collapse. Monasterio doesn’t limit his analysis to the study of mechanisms comprised only of sliding movements or only of rotation movements, but proposes a number of ‘mixed’ mechanisms that combine both types of movements, so that the motion in some of the joints is of sliding nature, and in other joints is of rotation nature. He will study ‘pure sliding’ mechanisms in the first chapter of the manuscript, ‘pure rotation’ mechanisms in the second and ‘mixed mechanisms’ in the third. In total, he studies 7 failure mechanisms, corresponding to the two pure motions and 5 mixed type motions (see Fig.1).

This article will focus on Monasterio’s analysis of ‘pure rotation’ mechanisms.

2.2 Material hypothesis and motion considerations

Before getting into the mathematics of the problem of vault stability, Monasterio defines in the *Introducción* a number of hypotheses about the mechanical properties of masonry that he will apply in his analysis:

(1) The density of the masonry is constant throughout the structure. The weights of the stones can therefore be substituted by the area, in planar structures, or by the volume, in three-dimensional structures.

(2) Compressive strength is not a limiting property for masonry vaulted structures. He backs this hypothesis on the fact that the compressive stresses in the stones are low compared to the strength of the material measured in experiments

(3) The tensile strength of the mortar is neglected. This is equivalent to saying that masonry cannot withstand tensile forces. Monasterio supports this saying that this strength is *‘inefficient or almost inexistent when the mortar is soft, that is, in newly-built arches’*.

(4) He acknowledges the existence of friction forces acting between the voussoirs and defends it must be taken into account in the analysis considering mechanisms with sliding motion.

The second and third hypotheses, when combined with the consideration that sliding between voussoirs does not occur (the case of ‘pure rotation’ mechanisms studied in the manuscript), form the basis of modern limit analysis of masonry structures, as stated by Heyman.

Monasterio also establishes a number of premises about the motion of vaults:

(1) He wants to determine the worst possible location of the joints that generate a mechanism in the vault, that is, the position at which the joint is most likely to form: if the arch is designed so that the joint cannot open there, it won’t be able to open anywhere. In order to obtain expressions that respond to a continuity law, Monasterio does not look at the exact position of the construction joints between the voussoirs, but considers the joints can open at any point along the arch.

(2) The joints in the vault will be considered to be radial.

3 ‘MOVIMIENTOS PUROS DE ROTACION’: PURE ROTATION MOTION

The second chapter of the manuscript is devoted to the discussion of pure rotation mechanisms in vaults subject to their own weight. As stated before, the hypotheses applied in this chapter coincide with those of modern limit analysis of vaulted structures.

3.1 General expression for a generic, non-symmetric, non-uniform arch

In his aim to obtain expressions of a general nature that can be applied to any arch, Monasterio commences his work studying the behaviour of a non-symmetric, non-uniform arch. He imagines the arch to fail by forming the mechanism with the minimum possible number of links, that is, splitting into three parts by four joints that open across the thickness of the arch, and around which the different pieces can only rotate. As can be seen on his *Fig.* 2^a in *Plate I* (Fig.2) these pieces are:

M”N”N’M’; *M’N’N’M’*; *M’N’N„M„*

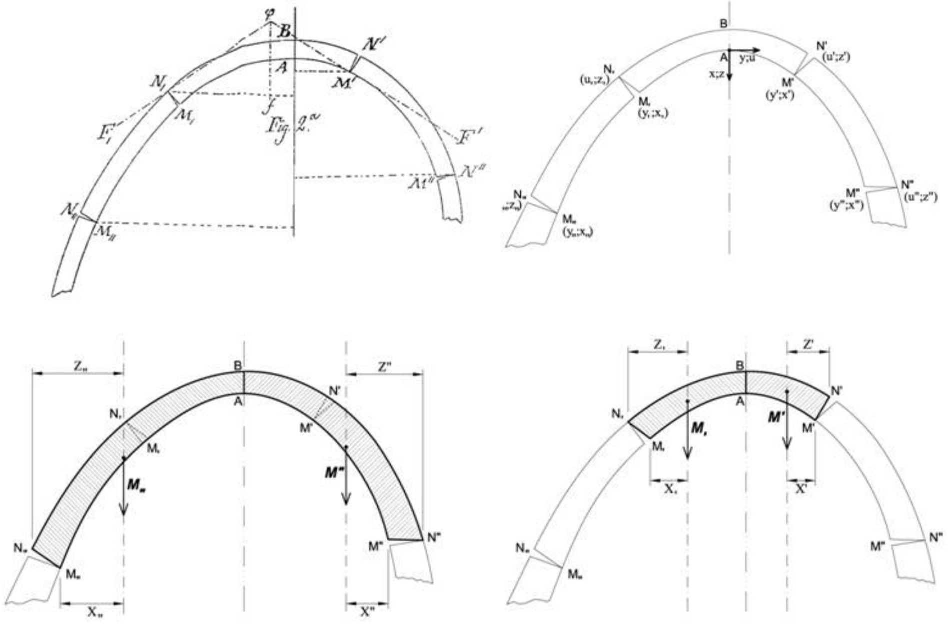

Figure 2 : Monasterio’s Manuscript Fig. 2ª, Plate I. Pure rotation failure mechanism.

For the weights acting in the analysis he uses a different nomenclature, represented in fig. 2. He considers an imaginary division at the keystone, understood as the highest point of the intrados curve, so that *M”* is the weight of the arch from the keystone to the lower-right joint (*M”N”*); *M„* is the weight from the keystone to the lower-left joint (*M„N„*); *M’* is the weight from the keystone to the top-right joint (*M’N’*); and *M’* is the weight from the keystone to the top-left joint (*M’N’*).

Monasterio defines the limit conditions that the divided arch must meet in order for the mechanism to be activated. The first of these conditions is that the weight of the middle piece must be divided into two equivalent forces, *F’* and *F’*, that meet on the vertical line through the centre of mass of the piece and must each go through the contact point of the adjacent joints, as shown by the dash-dot lines in *Fig. 2ª* (Fig.2). Here, he is effectively saying that the resultants of the weight of the middle piece, that is, the thrust line, must lie within the structure.

The next step is the formulation of equilibrium equations for the acting forces. The rotation motion will not take place if moment equilibrium is met at the contact points of the joints. After some algebraic manipulation, he obtains an equation in terms of the weights *M”*, *M’*, *M’* and *M„* and in terms of the coordinates of the lines of action of the said weights and of the corners of the three pieces the arch divides into:

$$a''M''Z'' - b''M'X' + c''M_{,}Z_{,} - d''M_{,,}X_{,,} \geq 0 \qquad (1)$$

where the terms *a”*, *b”*, *c”* and *d”* are equal to the following expressions:

$$\begin{aligned} a'' &= \left(x_{,,} - x'\right)\left(u_{,} + y'\right) + \left(x' - z_{,}\right)\left(y_{,,} + y'\right) \\ b'' &= \left(z'' - z_{,}\right)\left(y_{,,} - u_{,}\right) + \left(x_{,,} - z_{,}\right)\left(u'' + u_{,}\right) = a'' + c'' - d'' \\ c'' &= \left(z'' - x_{,}\right)\left(y_{,,} + y'\right) + \left(x_{,,} - x'\right)\left(u'' - y'\right) \\ d'' &= \left(z'' - x'\right)\left(u_{,} + y'\right) + \left(z_{,} - x'\right)\left(u'' - y'\right) \end{aligned} \qquad (2)$$

This is, in fact, the general expression Monasterio obtains for the initiation of the simplest rotation motion, and on it he will base the rest of the considerations included in the chapter.

There are, however, four unknowns in the equation. To solve it, Monasterio proposes to perform further algebraic operations, to express eq. 1 as:

$$\frac{x'-z_{,}}{d''}\left\{\frac{M''z''-M'x'}{z''-x'}-\frac{M'x'-M_{,}z_{,}}{x'-z_{,}}\right\}+\frac{x_{,,}-x'}{c''}\left\{\frac{M''z''-M'x'}{z''-x'}-\frac{M_{,,}X_{,,}-M'x'}{x_{,,}-x'}\right\}\geq 0 \qquad (3)$$

Then he fixes some of the variables to reduce the number of unknowns. He first defines *M”* as the piece of arch that goes from the keystone to the right-hand springing. The two terms of eq. 3 are then a function of *M’* and *M’*, and *M’* and *M„* respectively. Giving *M’* a certain value, the minimum values of the two terms can be found and compared. Finally, ‘*If the sum of these minimum values were always positive, under all different values of M’, whether the pieces M’, M”, M’ and M„ are taken from right to left as shown in Fig.2, or from left to right, we can be sure that the vault will be uncapable of acquiring pure rotation motion*’.

Since Eq. 1 is in fact an inequality, all algebraic operations performed must be carefully planned, for all multiplying factors or divisors must be positive. Monasterio checks this out in the manuscript demonstrating that variables *a”*, *b”*, *c”* and *d”* are positive for arches the stability of which needs to be studied (some of the variables become positive for very thick arches).

3.2 Symmetric vaults

The final section of the chapter is devoted to the analysis of symmetric vaults. First, the general expression for symmetric vaults is found, to be later adapted for the cases of a semicircular arch and a hemispherical dome.

In order to adapt his analysis for symmetric vaults, Monasterio proposes two different collapse mechanisms shown in *Figs. 12^a* and *13^a*, Plate I (Fig.3), corresponding respectively to the descent of the keystone and the overturning of the haunches. Indeed, the collapse of a symmetric vault will require the formation of at least 5 joints, as opposed to the 4 joints that appeared in the general, non-symmetric arch studied at the start of the chapter. These 5 joints will become 6 is the middle joint formed at the keystone splits into two symmetric joints located on either side of the original single one.

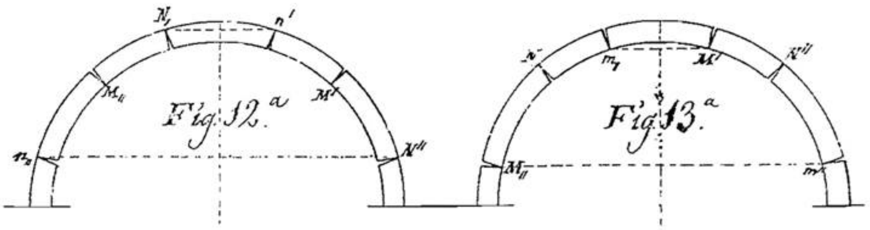

Figure 3 : Monasterio’s Manuscript *Figs. 12^a & 13^a*, Plate I. Possible pure rotation mechanisms for symmetric arches.

To account for both mechanisms, Monasterio substitutes independently *M„* by *M’* (*Fig.12^a*) and *M”* by *M’* (*Fig.13^a*) in eq. 1, obtaining the following expressions:

$$M''Z''\left(x'-z_{,}\right)-M'X'\left(z''-z_{,}\right)+M_{,}X_{,}\left(z''-x'\right)\geq 0 \qquad (4)$$

$$-M_{,,}X_{,,}\left(z_{,}-x'\right)+M_{,}Z_{,}\left(x_{,,}-x'\right)-M'X'\left(x_{,,}-z_{,}\right)\geq 0 \qquad (5)$$

In *Fig. 12^a* and *13^a*, the joints have been displaced with respect to *Fig.2^a*: two more joints have been introduced, and, in order for eq. 1 to be applicable, Monasterio has placed them so that the pre-existing joints preserve the same directions they had in the general arch of *Fig.2^a*. Known this, it is easy to understand that the symmetry conditions will be *M„ = M’* and *M” = M’* respectively (see Fig.4).

Monasterio is interested in demonstrating that, if a symmetric arch satisfies eqs. 4 and 5, it will also satisfy the general condition for preventing pure rotation mechanisms given in eq. 1. He considers all possible permutations of the sizes of the four pieces *M”*, *M’*, *M’* and *M„* and shows that, by substituting these permutations in turn into eq. 1, equivalent expressions to eqs. 4

and 5 are obtained. He concludes that this proves that, if eqs. 4 and 5 are satisfied, the general condition eq. 1 will also necessarily be satisfied: *if a symmetric vault cannot collapse under pure rotation motion by the formation of six joints, three of the joints located to one side of the keystone and in the same position as the other three in the opposite side, it won't collapse either with any other number of joints located differently on either side.*

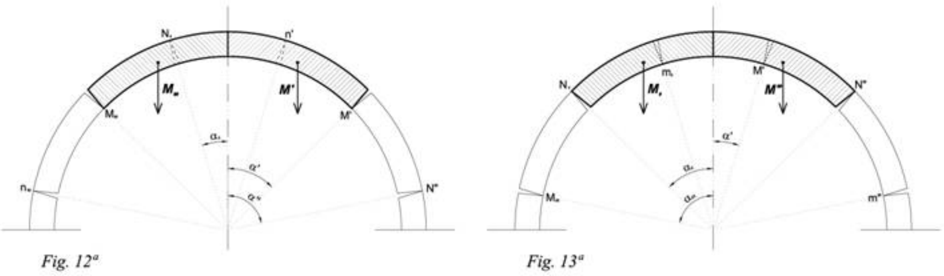

Figure 4 : Symmetry conditions imposed to obtain collapse mechanisms for symmetric arches.

Equations 8 and 9 can be simplified for certain cases that are commonly seen in practice. The simplification consists on making $M' = 0$ in eq. 4 and $M' = 0$ in eq. 5; then, dividing the first by $(z'' + K)(x' + K)$ and the second by $z'x_{,,}$, Monasterio obtains equations:

$$\frac{M''Z''}{z''+K} - \frac{M'X'}{x'+K} \geq 0 \tag{6}$$

$$\frac{M_{,}Z_{,}}{z_{,}} - \frac{M_{,,}X_{,,}}{x_{,,}} \geq 0 \tag{7}$$

This assumption implies that the two middle joints N' and n' drawn in *Fig.12ª*, and M' and m' in *Fig.13ª* will combine in one single joint at the axis of symmetry through the keystone (see Fig.5).

These simplified expressions should be used, with the exception of cases, for the mechanism of *Fig. 12ª*, when M'' being the minimum of $M''Z''/(z'' + K)$, and being M' the maximum of $M'Z'/(x' + K)$, the arch satisfies that $M'' < M'$ but does not satisfy eq. 6. Equivalently, for the mechanism in *Fig.13ª*, the simplified expression cannot be used when $M_{,,} < M'$ but eq. 10 is not satisfied.

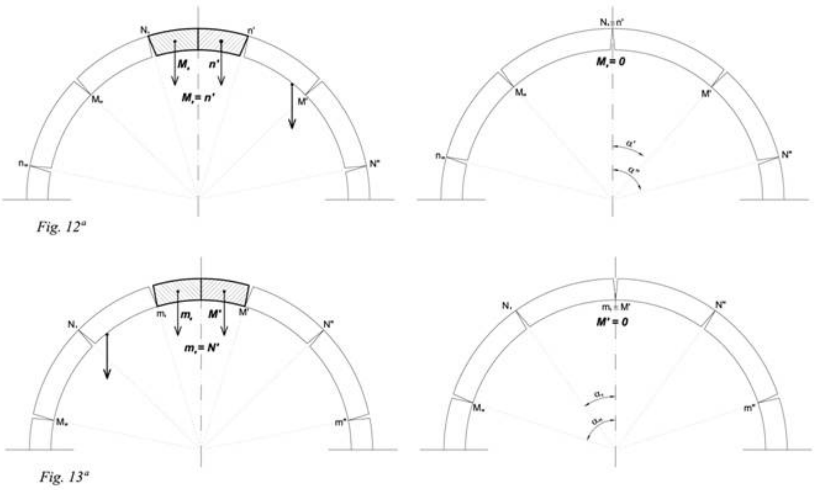

Figure 5 : Simplifications to mechanism of *Fig.12ª* and *13ª* (Fig.3).

3.3 *Constant-thickness semicircular arch*

Monasterio then applies this theory to a semicircular arch of constant thickness. The geometric parameters of this arch are known and can be substituted into eq. 6, obtaining[1]:

$$\frac{M''Z''}{K+z''} = 2\left(rK+\frac{1}{2}K^2\right)\cdot\frac{\frac{1}{2}\alpha''}{\tan\frac{1}{2}\alpha''}-\frac{1}{3}\left(\frac{3r^2K+3rK^2+K^3}{K+r}\right) \tag{8}$$

$$\frac{M'X'}{K+x'} = \frac{\left(rK+\frac{1}{2}K^2\right)r\alpha'\sin\alpha'-\left(1-\cos\alpha'\right)\left(r^2K+rK^2+\frac{1}{3}K^3\right)}{K+r-r\cos\alpha'} \tag{9}$$

In this particular case, Monasterio applies eq. 6 because by inspection he knows that the failure mode for a semicircular arch is that of the descent of the keystone. The results obtained confirm this observation and '*no arch can simultaneously acquire two motions as different as those represented in Figs.12th and 13th'*.

Monasterio wants to find the thickness *K*, expressed as a ratio of the intrados radius, *r*. He selects a thickness *K = 1/8 r* and substitutes it into the two expressions above, eqs. 8 and 9. To know whether the inequality given in eq. 6 is true –and thus, whether the arch is stable against rotation mechanisms-, the minimum value of the first term (eq. 8) and the maximum of the second term (eq. 9) must be found, the variables being the angles *α"* and *α'* respectively. If the minimum of the first term is greater than the maximum of the second term, pure rotation motion cannot start and the arch is stable under that mechanism.

The minimum of the first term was found to occur for *α" = 90°*. By trying whole values of *α'*, the maximum of the second term was found to be for *α' = 55°*.

Indeed, Monasterio found that an arch with a thickness *K* equal to an eighth of the internal radius is stable, while, if the thickness is reduced to *K = 1/9 r*, the pure rotation mechanism is activated. These results are, in fact, correct: in 1907 Milankovitch rigorously examined the problem and obtained a thickness *K = 0,1136 r = 1/8,80 r*, with the second joint opening at an angle *α' = 54°29'*.

Monasterio's analysis, though not mathematically rigorous, is correct from the static point of view, demonstrating his deep understanding of the problem of the mechanics of arches.

3.4 *Constant-thickness hemispherical dome*

The study of pure rotation motion in vaults concludes with the analysis of a hemispherical dome of constant thickness. By inspection, Monasterio determines that the failure mode of a hemispherical dome is that of *Fig.13ª* and eq. 10, the falling inwards of the haunches. He justifies this observation saying that '*the lower pieces are stronger or more heavily loaded than those located closer to the crown*'. This is, however, a mistake/a misperception, for a hemispherical dome, when failing by rotation motion only, fails by the fall of the keystone, with a total of 6 joints forming, three on each side of the keystone – exactly the failure mechanism shown in *Fig.12ª*. Monasterios system for selecting the failure mode leads to error: a more correct approach would have been to assess both mechanisms in *Figs.12ª* and *13ª*, and select that which requires the greater dome thickness.

Following the same procedure as for the semicircular arch, Monasterio substitutes the geometry of the hemispherical dome in eq. 10 to obtain:

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[1] A correction has been introduced in Equation 9: it appears in the manuscript without the *r* multiplying the first addend in the numerator of the second term. This error can be seen from the mathematical derivation of the expression, and without the *r* factor, the expressions would not be dimensionally coherent, for the first addend would have dimensions L^2, while the second would have L^3.

$$\frac{M_{''}X_{''}}{x_{''}} = \frac{{}^{1}\!/_{3}\left\{(r+K)^{3}-r^{3}\right\}\left(r-r\cos\alpha_{''}\right)\sin\alpha_{''} - {}^{1}\!/_{8}\left(\alpha_{''}-\sin\alpha_{''}\cos\alpha_{''}\right)\left\{(r+K)^{4}-r^{4}\right\}}{r-r\cos\alpha_{''}} \tag{10}$$

$$\frac{M_{'}Z_{'}}{z_{'}} = \frac{\frac{1}{3}\left\{(r+K)^{4}-r^{3}(r+K)\right\}\left(1-\cos\alpha_{'}\right)\sin\alpha_{'} - \frac{1}{8}\left(\alpha_{'}-\sin\alpha_{'}\cos\alpha_{'}\right)\left\{(r+K)^{4}-r^{4}\right\}}{r-(r+K)\cos\alpha_{'}} \tag{11}$$

The minimum possible thickness of the vault, according to Monasterio, lies between 1/23 and 1/24 of the radius of the intrados. He estimates values for the angles of the joints, *α” = 30°* and *α„ = 70°*, and, applying eq. 10 obtains the value of the minimum thickness referred above.
Despite the error above mentioned selecting the collapse mechanism, Monasterio manages to arrive at a result that is surprisingly close to the actual correct result: a thickness *K = 1/23.81*, and angle *α„ = 68°18’*.

4 CONCLUSIONS

The memoir by Monasterio presents the first application of Coulomb's theory to the study of the different collapse mechanisms of arches, predating in ten years that of Audoy. Besides, his discussion is remarkable for its generality and originality. In the present paper the usual hinge mechanism only has been discussed, but a publication of the whole memoire is in press. Monasterio was, also the first to obtain the correct limit thickness of a semicircular arch, locating correctly the haunch hinge at 55°. He was also the first to propose a calculation (in this case incorrect) of the limit thickness of a hemispherical dome. The memoir by Monasterio, though unpublished, marks a milestone in the history of masonry vault theory.

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