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Coherent combination of probabilistic outputs for group decision making: an algebraic approach

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Abstract Current decision support systems address domains that are heterogeneous in nature and becoming progressively larger. Such systems often require the input of expert judgement about a variety of different fields and an intensive computational power to produce the scores necessary to rank the available policies. Recently, integrating decision support systems have been introduced to enable a formal Bayesian multi-agent decision analysis to be distributed and consequently efficient. In such systems, where different panels of experts independently oversee disjoint but correlated vectors of variables, each expert group needs to deliver only certain summaries of the variables under their jurisdiction, derived from a conditional independence structure common to all panels, to properly derive an overall score for the available policies. Here we present an algebraic approach that makes this methodology feasible for a wide range of modelling contexts and that enables us to identify the summaries needed for such a combination of judgements. We are also able to demonstrate that coherence, in a sense we formalize here, is still guaranteed when panels only share a partial specification of their model with other panel members. We illustrate this algebraic approach by applying it to a specific class of Bayesian networks and demonstrate how we can use it to derive closed

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form formulae for the computations of the joint moments of variables that determine the score of different policies.

Keywords Bayesian networks · Integrating decision support systems · Polynomial algebra · Structural equation models

1 Introduction

Although still being refined, probabilistic decision support tools for single agents are now well developed and used in practice in a variety of domains. One of the most common probabilistic models for multivariate systems are Bayesian networks (BNs) (Jensen and Nielsen, 2013; Pearl, 1988) and their dynamic and object-oriented extensions (Koller and Pfeffer, 1997; Murphy, 2002). However, these are not the only frameworks around which probabilistic models have been built. Other well-established models comprise, among others, Bayesian hierarchical spatio-temporal models (Blangiardo and Cameletti, 2015), asymmetric probability trees (Smith and Anderson, 2008) and probabilistic emulators (Kennedy and O'Hagan, 2001).

However, the size and complexity of current applications often require that supporting systems consist of component modules which, encoding the judgements of panels of domain experts, describe a particular sub-domain of the overall system. In these contexts decision makers need a tool that can coherently paste together the outputs of each of these modules to provide a comprehensive picture of the whole process. Often in practice, because of both computational and methodological constraints, the modules' outputs end up being collated together in a simple, essentially deterministic way by transferring from one module to another a single vector of means about what might happen and hence effectively ignoring any associated uncertainty (French, 1997, 2011). However, such a naïve method can be very misleading and guide decision makers to choose a non-optimal course of action (see e.g. Leonelli and Smith, 2013, 2015).

Recently, integrating decision support systems (IDSSs) (Leonelli and Smith, 2015; Smith et al, 2015) have been defined to extend coherence requirements traditionally applied within a Bayesian decision support system for single agents so that these apply to this new multi-expert setting. IDSSs embed a methodology, similar to a standard Bayesian one, where decisions can be guaranteed to be coherent, i.e. expected utility maximising for some utility and probability distribution derived from individual but connected suites of models. Although some work has addressed the difficulties associated to the combination of expert judgments in multivariate systems (e.g. Faria and Smith, 1997; Farr et al, 2019), none of these formally took into account the heterogeneity of the domain to be modelled. Conversely, some informal protocols to guide this probabilistic integration have been discussed only in the context of BN models (Johnson and Mengersen, 2012; Mahoney and Laskey, 1996), but these do not formally investigate the conditions under which decision support can be expected utility maximizing in such heterogeneous settings.

In Smith et al (2015) we focused on the inferential full-distributional difficulties associated to this integration. However, a formal Bayesian decision analysis is based on the maximization of an expected utility (EU) function and this often only depends on some simple summaries of key output variables, for example a few low order moments. By requesting from the relevant panels only the value of these expectations, as long as they come from an agreed type of utility function, the implementation of an IDSS can become orders of magnitude more manageable. Panels then just need to communicate a few summaries of their analysis: a trivial and fast task to perform within most inferential systems. In these cases real time decision support is thus feasible even when the system is huge.

The EUs of such an IDSS are often polynomials whose indeterminates are functions of the panels' delivered summaries, as exemplified in our application reported in Section 8. However, a polynomial form for the EU would be entertained for most utility functions and the wide array of probabilistic models which have a polynomial representation (see e.g. Brandherm and Jameson, 2004; Castillo et al, 1997; Darwiche, 2003; Görgen et al, 2015; Leonelli et al, 2017, among others). This polynomial structure enables us to identify new separation conditions, often implicit in standard conditional independence over the parameters of certain graphical models (Freeman and Smith, 2011; Spiegelhalter and Lauritzen, 1990) and milder than those of Smith et al (2015), sufficient to guarantee that the recommendations of an IDSS are EU maximizing and computed from utility and probability functions delivered by individual but connected suites of models. In general such conditions are imposed to guarantee specific factorizations of the EU functions and consist in writing the expectation of the product of some specific summaries as the product of the expectations.

Under the conditions derived above, we develop new propagation algorithms for BNs, here called *algebraic substitutions*, for the distributed computations of an IDSS EU scores. These generalize the theory of the computation of moments of decomposable functions (Cowell et al, 1999; Nilsson, 2001), which correspond to an additive decomposition of some function defined over the variables of a BN, to multilinear ones. Both decomposable and multilinear functions are useful when the aim is to estimate various functionals of the variables of a BN, for instance a joint utility function. Algebraic substitutions mirror the recursions of Lauritzen (1992) for the computation of the first two moments of chain graph models. Here, focusing only on specific BN models, we are able to explicitly compute any joint moment and provide an intuitive graphical interpretation of the associated propagation rules. Although algebraic substitutions could be used in conjunction with the updating of the conditional probabilities over the variables of the BN as new information/data become available (using e.g. the methods of Cooper and Yoo, 1999), such issues are investigated in Smith et al (2015) in details.

Importantly, the recognition of the polynomial nature of EUs also enables us to analyze efficiently as well as exactly even large problems using software for symbolic manipulations (or *computer algebra* software), e.g. Mathematica

(Wolfram Research, Inc., 2017). Assuming the panels are able to deliver a vector of required summaries from the complex probabilistic model they plan to use, the software is then capable of combining them using algebraic substitutions to compute the associated EU scores almost instantaneously and in real-time to evaluate the candidate policies available to a decision centre. This is a critical property of any decision support system and in Section 8 we give an illustration of how this can be achieved with computer algebra software.

The paper is structured as follows. Section 2 reviews the theory of IDSSs, whilst Section 3 introduces one domain of application where we have found it necessary to knit together a suite of models as well as our running example. A new polynomial representation of EUs is introduced in Section 4 and, based on this formalism, Section 5 defines our new separation conditions. Section 6 specifies the cases when an IDSS is EU maximizing. Instances of these results are then reported for the critical class of BN models in Section 7. An operational implementation of our results in a computer algebra system are reported in Section 8. A discussion wraps up the paper.

2 How an IDSS works

Although the decomposition of a complex system into connected but separated components overseen by different panels of experts may seem reasonable in most cases (in particular in the domains of food security (Barons et al, 2017) and nuclear emergency management (Leonelli and Smith, 2013, 2015), which have motivated the methodological developments we present below, this is the case), various conditions need to be satisfied for an IDSS to be justifiable. More specifically it can be argued that an IDSS requires the following to hold:

- the decision centre responsible for the implementation of any policy needs to consist of individuals who act collaboratively and strive to behave as a single coherent unit would. In the food security (Barons et al, 2017) and nuclear emergency management (Leonelli and Smith, 2013) applications this condition was broadly met. We suppose the centre consists of m panels of experts denoted by $G_1, \dots, G_m$;
- there must be a consensus about the policies d that could be scrutinized and eventually implemented by the centre. In other words, all individuals in the centre must agree on a set $\mathbb{D}$ of policies whose efficacy might be examined by the IDSS. The choice of $\mathbb{D}$ is usually resolved using decision conferencing (French et al, 2009) across panel representatives, users and stakeholders. We refer to this condition as *policy consensus*;
- there must also be a consensus about the appropriate utility structure underlying a set of agreed attributes against which the efficacy of any policy is evaluated. So all individuals in the centre need to agree on the class $\mathbb{U}$ of utility functions supported by the IDSS. For instance, this consensus might be that the centre's utility function has utility independent attributes (Keeney and Raiffa, 1976). Again the choice of $\mathbb{U}$ is often re-

solved through decision conferencing. We refer to this condition as *utility consensus*;
- consensus also needs to be found about an overarching description of the dynamics driving the process. We assume that all panellists make their inferences in a parametric setting where a random vector $\boldsymbol{Y}$ is parametrised by a vector $\boldsymbol{\theta}$. The vector $\boldsymbol{Y}$ defines the variables of the process whilst $\boldsymbol{\theta}$ is the parameter vector on which inference is made. Then such a consensus consists of an agreement of all involved on the variables $\boldsymbol{Y}$, where, for each policy $d \in \mathbb{D}$, each utility function $u \in \mathbb{U}$ is a function of $\boldsymbol{Y}$, together with a set of qualitative statements about the dependence between various functions of $\boldsymbol{Y}$ and $\boldsymbol{\theta}$. This can take a variety of forms depending on the domain of application, but in general it consists of a conditional independence structure over $\boldsymbol{Y}$ and $\boldsymbol{\theta}$. In this paper we mainly focus on dependence structures represented by BNs, although our methods apply equally well to other frameworks (Smith et al, 2015). We refer to this condition as *structural consensus.*

Notice that in the case policy, utility or structural consensus are not reached, this would imply that the IDSS would require summaries that panels may not be willing or capable of delivering, thus compromising the whole integrity of the system. As an example, suppose one panel believes that there are additional policies available for scrutiny and that such decisions have a direct influence on variables under the jurisdiction of other panels. Then these other panels may not have the capabilities or the interest in delivering the appropriate summaries associated to such decisions that they are not contemplating.

The union of the policy, utility and structural consensus is called the *common-knowledge class* (CK-class) and describes the agreement of all individuals on the components of the system and their relationships with each other[1]. The CK-class defines the *qualitative* structure of the domain investigated and therefore more easily provides a framework for the group's agreement (Smith, 1996). Protocols to guide the construction of the sets $\mathbb{D}$ and $\mathbb{U}$, and the identification of an overarching probabilistic model for the structural consensus have been recently defined (Barons et al, 2017).

Given this overarching qualitative structure has been agreed by the centre and represented by the CK-class, then an agreement on how to populate this class with *quantitative* statements must be found. To this end, we assume the following condition holds:

- the centre must find a consensus about who is expert about what. In a formal sense, this implies that all panellists are prepared to adopt the beliefs of the designated expert panel in a specific sub-domain of the process as their own.

[1] The CK-class is similar in nature to the SupraBayesian approach in standard Bayesian combination of subjective distributions approach (French, 2011). Here we use the terminology CK-class to emphasize that only some specific information needs to be shared and agreed upon by the different panels of experts.

Thus in an IDSS beliefs' specifications are delegated to the most informed panel. Each panel then, given a CK-class, individually delivers the necessary quantities for the computation of expected utilities concerning the variables under their jurisdiction. However, there is in general no guarantee that the individual beliefs of the panels can be combined to give a probabilistic coherent overall picture of the process. For the purposes of a formal Bayesian decision analysis an IDSS needs to satisfy the following property.

Definition 1 An IDSS is said to be *adequate* for a CK-class if it can unambigously calculate the expected utility score of any policy $d \in \mathbb{D}$ under any utility function $u \in \mathbb{U}$ and any beliefs of the panels $G_1, \dots, G_m$.

It is vital for an IDSS to be adequate since otherwise it could not produce a ranking of the available policies (Keeney and Raiffa, 1976) and would therefore not be of any help to the decision centre for implementing and justifying any policy choice.

3 UK household food security

Food security exists when all people, at all times, have physical and economic access to sufficient, safe and nutritious food that meets their dietary needs and food preferences for an active and healthy life (FAO, 1996). Food security, once thought to be a problem confined to low-income countries, is increasingly being recognised as a matter of concern in wealthy nations like the UK (Dowler and Lambie-Mumford, 2015). At a country level the UK is among the most food secure in the world, but at household level the number of individuals with limited or uncertain availability of nutritionally adequate and safe foods is increasing rapidly (Loopstra et al, 2015). At first glance, UK household food security may seem to be a simple case of demand and supply. On closer inspection though, and as illustrated in the next paragraph, the food system is highly complex, especially from the viewpoint of policymakers, who endeavour to intervene on the system in order to provoke specific ameliorating responses, for instance to ensure that healthy food is always available in supermarkets or to guarantee that every household has enough resources to access healthy food (Dowler and Lambie-Mumford, 2015; Drewnowski and Specter, 2004).

The food system is global, multifaceted and influenced by a huge number of public and private actions and uncontrolled factors such as weather, pests and disease. This leads to a great deal of uncertainty about the effectiveness of any one policy. One of the authors has been involved, in partnership with Warwickshire County Council, UK, in the development of an IDSS to support decision-making around household-level food poverty. An overall description of the highly heterogeneous food system requires the judgements from different panels of experts in diverse disciplines including insights about factors elevating the risk to food security of households (from sociologists and local authorities), judgements about the effects of malnutrition on the population (from doctors and nutritionists), estimates of the availability of food in supermarkets and other outlets (from supply chain experts) and forecasts of the yield of crops in a particular season (from crop experts).

Unless properly structured, this expert information is liable to conflict since two or more panels can sometimes deliver contradicting expert judgements about a shared random variable. If such contradictions are admitted then the system's adequacy is obviously threatened and its outputs become compromised. For instance, both estimates of cost of oil and weather forecasts affect food production, food transport and the ability of households to access food. Suppose these latter variables are under the jurisdiction of different panels. Then the distributions of food production, transport and access independently delivered by the different panels should be conditioned on the same estimates of oil cost and weather forecasts and not contradicting ones. Otherwise how could be the recommendations of the integrating system ever be justifiable?

Whilst numerous systems to model aspects of the composite process exist, such as for supermarket locations and food demand forecasting (Efendigil et al, 2009; Hernandez and Bennison, 2000), the complex problem of developing a shared methodology to guide the accommodation of diverse expertise and that provide enough information to evaluate the efficacy of various policies designed to address food poverty issues has been attempted only recently (Barons et al, 2015; Smith et al, 2015).

3.1 IDSS for food security

After a series of decision conferences with local authorities from Warwickshire County Council, stakeholders and potential decision makers, Barons et al (2017) identified three areas that are impacted by poor household food security: health (Y_1), educational attainment (Y_2) and social cohesion (Y_3). Of course the cost (Y_4) associated to the enactment of any policy is deemed relevant in this domain. Measurable indices were developed for each of these areas - for instance, educational attainment is assessed by the percentage of pupils not failing a combination of UK school examinations. Further details about the form of the attributes are beyond the scope of this paper and we refer to Barons et al (2017) for a discussion of these (see also Leonelli and Smith, 2017). For the purposes of this paper, we assume that all these indexes are percentages (for instance cost can be thought of the percentage of the budget available assigned to food security issues). However, for our methodology to apply this does not have to be the case.

Notice that of course in a reliable description of the food system any decision support system needs to account for the probabilistic dependence over a much larger vector of variables. But for the illustrative purposes of this example, we assume the decision centre agrees that these four random variables provide an overall, sufficient description of the household food system. Even in such a simple example we can highlight the heterogeneity of the food system

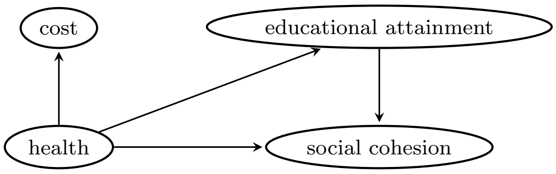

Fig. 1 BN representing the relationship between the four attributes in the food insecurity example.

and the consequent need of an IDSS. So for instance beliefs about the health attribute are delivered by doctors and public health experts; educational attainment is under the jurisdiction of school representatives and teachers; social unrest is overseen by sociologists, whilst judgements about costs are given by politicians and policymakers.

We consider a decision space $\mathbb{D}$ comprising of three possible policies: either an increase (d_0), a decrease (d_1) or not a change (d_2) of the number of pupils eligible for free school meals in Warwickshire. The UK government has already implemented this type of policy to facilitate access to nutritional food for pupils, since evidence seems to point towards an improvement of development and social skills of young children that eat a healthy meal together at lunchtime (Kitchen et al, 2013). We suppose henceforth that the decision centre, consisting of local authorities and stakeholders, agrees to consider only these three policies.

The utility consensus might correspond to an agreement between all panels G_i of a specific utility factorization over these four attributes. For instance, letting $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, where y_i is an instantiation of Y_i, the centre might find an agreement that the utility function factorizes additively. Specifically,

$$u(d, \boldsymbol{y}) = k_1u_1(y_1, d) + k_2u_2(y_2, d) + k_3u_3(y_3, d) + k_4u_4(y_4, d), \quad (1)$$

where the criterion weights $k_i \in (0, 1)$ are agreed by all panels G_i and $d \in \mathbb{D}$. Given this agreed factorization, the specific form of the functions $u_i(y_i)$ is then elicited by the appropriate expert panel.

The structural consensus of an IDSS, agreed upon by the various panels of experts, with these four random variables then consists of a conditional independence structure that we suppose here to be depicted by the BN in Fig. 1. This states that given different levels of the health attribute, the associated costs are independent of both educational attainment and social cohesion.

In the following sections we illustrate our methodology using this simple IDSS for household food security.

4 An algebraic description of IDSSs

We start by giving a polynomial description of the EUs of an IDSS. Consider a random vector $\boldsymbol{Y} = (\boldsymbol{Y}_i)_{i\in[m]}$, $[m] = \{1, \ldots, m\}$, where a subvector $\boldsymbol{Y}_i$ of $\boldsymbol{Y}$ is under the jurisdiction of a panel of experts G_i, $i \in [m]$. Let $\boldsymbol{y} \in \mathbb{Y}$ and $\boldsymbol{y}_i \in \mathbb{Y}_i$ be instantiations of $\boldsymbol{Y}$ and $\boldsymbol{Y}_i$, respectively. Assume each panel of experts delivers beliefs about $\boldsymbol{\theta}_i$, the parameter of the density f_i over $\boldsymbol{Y}_i \mid (\boldsymbol{\theta}_i, d)$, where $d \in \mathbb{D}$ is one of the available policies in the decision space $\mathbb{D}$. Suppose $\boldsymbol{\theta}_i$ takes values in $\boldsymbol{\Theta}_i$ and let $\boldsymbol{\theta} = (\boldsymbol{\theta}_i)_{i\in[m]}$ take values in $\boldsymbol{\Theta}$. Let f, π_i and π denote densities over $\boldsymbol{Y} \mid (\boldsymbol{\theta}, d)$, $\boldsymbol{\theta}_i \mid d$ and $\boldsymbol{\theta} \mid d$, respectively.

The IDSS processes the panels' judgements in order to calculate various statistics of an *attribute vector*, usually some function of $\boldsymbol{Y}$. For simplicity and with no loss of generality we assume in this paper that attributes coincide with $\boldsymbol{Y}$. For the purpose of a formal Bayesian analysis the IDSS computes the set of *EU scores* $\{\bar{u}(d) : d \in \mathcal{D}\}$ as a function of both the utility function $u(\boldsymbol{y}, d)$ agreed by all panels G_i, $i \in [m]$, and the probability statements of the individual panels. The IDSS would then recommend to follow the policy d^* with the highest EU score, $\bar{u}(d^*)$, where the EU is computed as

$$\bar{u}(d) = \int_{\boldsymbol{\Theta}} \bar{u}(d \mid \boldsymbol{\theta})\pi(\boldsymbol{\theta} \mid d)\mathrm{d}\boldsymbol{\theta},$$

and

$$\bar{u}(d \mid \boldsymbol{\theta}) = \int_{\mathbb{Y}} u(\boldsymbol{y}, d) f(\boldsymbol{y} \mid \boldsymbol{\theta}, d)\mathrm{d}\boldsymbol{y},$$

is the *conditional EU*.

By approaching the theory of IDSSs from an algebraic viewpoint, we are able to identify the necessary panels' summaries and the required assumptions for adequacy. In order to do this we first need to define the EU polynomials.

Let $\boldsymbol{\lambda}_i(\boldsymbol{\theta}_i, d) = (\lambda_{ji}(\boldsymbol{\theta}_i, d))_{j\in[s_i]}$ be a vector of length s_i including the s_i summaries panel G_i is required to deliver and $\lambda_{0i}(\boldsymbol{\theta}_i, d) = 1$, for $i \in [m]$. These will be the indeterminates of the EU polynomials. Let $[s_i]^0 = [s_i] \cup \{0\}$ and $B = \times_{i\in[m]}[s_i]^0$. Therefore a $\boldsymbol{b} = (b_i)_{i\in[m]} \in B$ is a vector of length m whose entry b_i indicates one of the summaries delivered by panel G_i. For a given $\boldsymbol{b} = (b_i)_{i\in[m]}$ and $j \in [s_i]$, define $b_{j,i} = 0$ if $j \neq b_i$, $b_{j,i} = 1$ if $j = b_i$ and $b_{0,i} = 1$, for $i \in [m]$. It follows that $b_{j,i}$ is not zero if and only if either $j = 0$ or j equals the i-th entry of $\boldsymbol{b}$. The following example illustrates the construction of the variables $b_{i,j}$.

Example 1 Let $m = 2$, $s_1 = s_2 = 1$, i.e. there are two panels each delivering one summary only. Then $B = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. For $\boldsymbol{b} = (0, 1) \in B$ we have that $b_{0,1} = 1$, $b_{1,1} = 0$, $b_{0,2} = 1$ and $b_{1,2} = 1$.

Definition 2 The conditional EU $\bar{u}(d \mid \boldsymbol{\theta})$ of an IDSS is called *algebraic* if, for each $d \in \mathcal{D}$, $\bar{u}(d \mid \boldsymbol{\theta})$ is a square-free polynomial of the $\lambda_{ji}(\boldsymbol{\theta}_i, d)$, $i \in [m]$, $j \in [s_i]^0$, i.e. if it can be written as

$$\bar{u}(d \mid \boldsymbol{\theta}) = \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \lambda_{\boldsymbol{b}}(\boldsymbol{\theta}, d), \tag{2}$$

with $k_{\boldsymbol{b}} \in \mathbb{R}$ and

$$\lambda_{\boldsymbol{b}}(\boldsymbol{\theta}, d) = \prod_{i\in[m]} \prod_{j\in[s_i]^0} \lambda_{ji}(\boldsymbol{\theta}_i, d)^{b_{j,i}}.$$

Thus the conditional EU in equation (2) is a polynomial where each $\lambda_{\boldsymbol{b}}$ is a monomial having at most one term not unity delivered by each panel and $k_{\boldsymbol{b}}$ is a weight. By summing over all possible $\boldsymbol{b} \in B$ in equation (2) we are considering every possible combination of the summaries delivered by the panels. The following two examples illustrate the polynomial form of the conditional EU $\bar{u}(d|\boldsymbol{\theta})$ and the construction of the vectors $\boldsymbol{\lambda}_i(\boldsymbol{\theta}_i, d)$.

Example 2 Let the CK-class specify that $\boldsymbol{Y} = (Y_i)_{i\in[m]}$, where each variable Y_i is binary and overseen by panel G_i. Assume that for all policies $d \in \mathcal{D}$, $\theta_i = \mathbb{P}(Y_i|\theta_i, d)$, $\boldsymbol{\theta} = (\theta_i)_{i\in[m]}$, and that the CK-class includes the belief that $\perp\!\!\!\perp_{i\in[m]} Y_i|\boldsymbol{\theta}, d$, where $\perp\!\!\!\perp$ denotes conditional independence (Dawid, 1979). Suppose the utility consensus consists of a utility factorization of the form

$$u(\boldsymbol{y}) = u(y_1, \dots, y_m) = \sum_{i\in[m]} k_i y_i + \sum_{i\in[m]} \sum_{i<j\leq m} k_{ij} y_i y_j,$$

where k_i and k_{ij} are jointly agreed criterion weights (French et al, 2009). With no further assumption the conditional EU can be written as

$$\bar{u}(d|\boldsymbol{\theta}) = \sum_{i\in[m]} k_i \lambda_{1i}(\theta_i, d) + \sum_{i\in[m]} \sum_{i<j\leq m} k_{ij} \lambda_{1i}(\theta_i, d) \lambda_{1j}(\theta_i, d), \tag{3}$$

where $\lambda_{1i}(\theta_i, d) = \theta_i$. Thus equation (3) is an algebraic conditional EU and $\boldsymbol{\lambda}_i(\boldsymbol{\theta}_i, d) = (\theta_i)$.

Example 3 Consider the CK-class for the food security example introduced in Section 3.1. Suppose each panel independently decides to model the utility function under their jurisdiction in equation (1) using a simple linear function. Explicitly, suppose that $u_i(y_i, d) = y_i$ for $i \in [3]$ and $u_4(y_4, d) = -y_4$ (it is less desirable to have higher costs). So the IDSS utility function simplifies to:

$$u(\boldsymbol{y}, d) = k_1 y_1 + k_2 y_2 + k_3 y_3 - k_4 y_4$$

Further suppose that each panel decides to model the variable under its jurisdiction as a simple linear regression over the parent variables of the BN in Figure 1. Formally:

$$\begin{array}{ll} Y_4 = \theta_{04} + \theta_{14} Y_1 + \varepsilon_1, & Y_3 = \theta_{03} + \theta_{13} Y_1 + \theta_{23} Y_2 + \varepsilon_3, \\ Y_2 = \theta_{02} + \theta_{12} Y_1 + \varepsilon_2, & Y_1 = \theta_{01} + \varepsilon_1, \end{array} \tag{4}$$

where $\theta_{ij} \in \mathbb{R}$ and ε_i is a mean-zero normal error, i.e. $\varepsilon_i \sim \mathcal{N}(0, \sigma_i^2)$. Notice that the variables' definition in equation (4) is compatible with the structural consensus of the CK-class and represented by the BN in Figure 1 (see Section 7 below for more details). Using standard properties of conditional moments (see e.g. Brillinger, 1969), by leaving implicit the dependence on d, the conditional EU can be written as

$$\begin{aligned} \bar{u}(d|\boldsymbol{\theta}) = k_1\theta_{01} + k_2\theta_{02} + k_2\theta_{01}\theta_{12} + k_3\theta_{03} + k_3\theta_{13}\theta_{01} + \\ k_3\theta_{01}\theta_{12}\theta_{23} + k_4\theta_{04} + k_4\theta_{01}\theta_{14} \end{aligned} \tag{5}$$

The conditional EU is therefore algebraic and

$$\begin{aligned} \boldsymbol{\lambda}_4(\boldsymbol{\theta}_4, d) &= (\theta_{04}, \theta_{14}), & \boldsymbol{\lambda}_3(\boldsymbol{\theta}_3, d) &= (\theta_{03}, \theta_{13}, \theta_{23}), \\ \boldsymbol{\lambda}_2(\boldsymbol{\theta}_2, d) &= (\theta_{02}, \theta_{12}), & \boldsymbol{\lambda}_1(\boldsymbol{\theta}_1, d) &= (\theta_{01}). \end{aligned}$$

So the panels only need to concern about the baseline value of the variable under their jurisdiction (measured by the parameters θ_{0i}) and the strength of dependence with other relevant variables (measured by the parameters θ_{ij} for $i \neq 0$).

The following property, that we refer to as *score separability*, is introduced to guarantee that an IDSS is adequate.

Definition 3 Let $\mu_{ji}(d) = \mathbb{E}(\lambda_{ji}(\boldsymbol{\theta}_i, d))$, for $i \in [m]$ and $j \in [s_i]$. We call an IDSS *score separable* if, in the notation above, all panellists agree that, for all policies $d \in \mathbb{D}$ and all indices $\boldsymbol{b} \in B$ such that $k_{\boldsymbol{b}} \neq 0$,

$$\mathbb{E}(\lambda_{\boldsymbol{b}}(\boldsymbol{\theta}, d)) = \prod_{i \in [m]} \prod_{j \in [s_i]^0} \mu_{ji}(d).$$

The condition of score separability is implied by the standard, almost always assumed conditional independences over the parameters of various graphical models (Freeman and Smith, 2011; Spiegelhalter and Lauritzen, 1990). Therefore in practice score separability is almost always assumed.

A score separable IDSS can then determine the EU score of any policy $d \in \mathbb{D}$ from the summaries $\mu_{ij}(d)$ individually delivered by the panels, $i \in [m]$, $j \in [s_i]$. This implies adequacy as formalized in Lemma 1 below. For every $d \in \mathcal{D}$, let $\boldsymbol{\mu}_i(d) = (\mu_{ji}(d))_{j \in [s_i]}$.

Lemma 1 *Suppose G_i delivers its vectors of expectations $\boldsymbol{\mu}_i(d)$, $i \in [m]$, $d \in \mathcal{D}$. For an algebraic conditional EU, if the IDSS is score separable then it is adequate.*

The proof of this result follows from the definition of algebraic conditional EU in equation (2) and the definition of score separability.

Example 4 (Example 2 continued) From equation (3) we can deduce that the score separability condition corresponds to the factorization of the expectations $\mathbb{E}(\theta_i\theta_j)$, $i, j \in [m]$, $i \neq j$, into $\mathbb{E}(\theta_i)\mathbb{E}(\theta_j)$.

Example 5 (Example 3 continued) Score separability corresponds to the conditions

$$\begin{aligned} &\mathbb{E}(\theta_{01}\theta_{12}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12}) \quad \mathbb{E}(\theta_{01}\theta_{13}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{13}) \\ &\mathbb{E}(\theta_{01}\theta_{14}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{14}) \quad \mathbb{E}(\theta_{01}\theta_{12}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{23}). \end{aligned} \tag{6}$$

In particular this means that the panel overseeing Y_1 (health) needs to agree with all other panels that the expectation of the baseline θ_{01} factorizes with respect to the parameter measuring the strength of dependence between health and all other variables, i.e. θ_{ij}, $i \neq 0$. Furthermore the panels overseeing health, educational attainment and social cohesion need to be ready to accept that $\mathbb{E}(\theta_{01}\theta_{12}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{23})$. Such assumptions would be met if the panels believed that these parameters are all independent of each other.

5 Moment and quasi independence

Lemma 1 shows that adequacy is guaranteed whenever score separability holds for algebraic conditional EUs. This implies that the expectation of certain functions of the panels' parameters separate appropriately. We first introduce conditions that ensure this type of separability and then in Section 6 identify classes of models that give rise to algebraic conditional EUs.

Definition 4 Let $\bar{u}(d|\boldsymbol{\theta})$ be the algebraic conditional EU of an IDSS. An IDSS is called *quasi independent* if

$$\mathbb{E}(\bar{u}(d|\boldsymbol{\theta})) = \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \prod_{i\in[m]} \prod_{j\in[s_i]^0} \mathbb{E}(\lambda_{ji}(\boldsymbol{\theta}_i, d)^{b_{j,i}}).$$

This condition requires the expectation of the product of certain functions of the parameters overseen by different panels to be equal to the product of the individual expectations. Notice that if quasi-independence did not hold, then the IDSS would not have enough information to compute the EU scores from the individual panel summaries $\boldsymbol{\mu}_i(d)$ since it would require an estimate of the strength of dependence between parameters overseen by different panels.

Often the λ_{ji}, $i \in [m]$, $j \in [s_i]$, are monomial functions of the panels' parameters. This was the case in Examples 2 and 3 above. It is therefore helpful to introduce the following independence condition specific for monomial functions. Let $<_{lex}$ denote a lexicographic order (Cox et al, 2007).

Definition 5 Let $\boldsymbol{\theta} = (\theta_i)_{i\in[m]} \in \mathbb{R}^m$ be a parameter vector and $\boldsymbol{c} = (c_i)_{i\in[n]} \in \mathbb{Z}^n_{\geq 0}$. We say that $\boldsymbol{\theta}$ satisfies *moment independence* of order $\boldsymbol{c}$ if for any $\boldsymbol{a} = (a_i)_{i\in[n]} <_{lex} \boldsymbol{c}$, $\boldsymbol{a} \in \mathbb{Z}^n_{\geq 0}$, and letting $\boldsymbol{\theta}^{\boldsymbol{a}} = \theta_1^{a_1} \cdots \theta_n^{a_n}$, it holds

$$\mathbb{E}\left(\boldsymbol{\theta}^{\boldsymbol{a}}\right) = \prod_{i\in[n]} \mathbb{E}\left(\theta_i^{a_i}\right).$$

Example 6 In Example 5 we say that score separability holds for the conditional EU in equation (5) if e.g. $\mathbb{E}(\theta_{01}\theta_{12}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})$ and $\mathbb{E}(\theta_{01}\theta_{12}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{23})$. Both these requirements correspond to a moment independence of order $\boldsymbol{c}$, where $\boldsymbol{c}$ is a vector including only ones or zeros.

It is generally well known that standard probabilistic independence only guarantees that the first moment of a product can be written as the product of the moments. Separations for higher orders are implied by standard independence only through a cumulant parametrization, where the cumulant generating function for a product of independent random variables (defined as a random sum of independent realizations) is the composition of the respective cumulant generating functions (Brillinger, 1969).

For the purpose of decision support it is helpful to study moments, since expected utilities often formally depend on these. Consider for instance two parameters θ_1 and θ_2. Assume a conditional EU is equal to $\theta_1^2\theta_2^2$ and that a moment independence of order $(2,2)$ holds. Then

$$\begin{aligned}\mathbb{E}\left(\theta_1^2\theta_2^2\right) &= \mathbb{E}\left(\theta_1^2\right)\mathbb{E}\left(\theta_2^2\right)\\ &= \mathbb{E}(\theta_1)^2\mathbb{E}(\theta_2)^2 + \mathbb{E}(\theta_1)^2\mathbb{V}(\theta_2) + \mathbb{E}(\theta_2)^2\mathbb{V}(\theta_1) + \mathbb{V}(\theta_1)\mathbb{V}(\theta_2). \quad (7)\end{aligned}$$

The same expression is obtained when using sequentially the tower rule of expectations and the law of total variance under the assumption of independence of the two parameters above (Brillinger, 1969). Therefore, the expression obtained under moment independence is reasonable and coincides with the one implied by the independence of θ_1 and θ_2. However the condition we need for equation (7) to hold *does not require* θ_1 and θ_2 to be independent.

6 Adequate combinations of probabilistic outputs

Given the above definitions of new independence concepts tailored to IDSSs, we can now study situations where these can be shown to be adequate and therefore provide a coherent, operational support tool to decision centres.

Proposition 1 *Let $\bar{u}(d|\boldsymbol{\theta})$ be an algebraic conditional EU of a quasi independent IDSS. The IDSS is adequate if panel G_i delivers the vectors of expectations $\boldsymbol{\mu}_i(d)$, for all $i \in [m]$ and $d \in \mathcal{D}$.*

This result follows by noting that quasi independence implies score separability since

$$\bar{u}(d) = \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \prod_{i\in[m]} \prod_{j\in[s_i]^0} \mathbb{E}(\lambda_{ji}(\boldsymbol{\theta}_i, d)^{b_{j,i}}) = \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \prod_{i\in[m]} \prod_{j\in[s_i]^0} \mu_{ji}(d).$$

Assuming the conditional EU is a polynomial in the panels' parameters, under a specific moment independence assumption we have a more operational result.

Corollary 1 *Let $\bar{u}(d|\boldsymbol{\theta})$ be an algebraic conditional EU of an IDSS, $\boldsymbol{\theta}_i = (\theta_{ji})_{j\in[s_i]}$ and $\lambda_{ji}(\boldsymbol{\theta}_i, d) = \boldsymbol{\theta}_i^{\boldsymbol{a}_{ji}}$, with $\boldsymbol{a}_{ji} \in \mathbb{Z}_{\geq 0}^{s_i}$, $i \in [m]$, $j \in [s_i]$. Let $\boldsymbol{a}_i^* = (a_{ji}^*)_{j\in[s_i]}$, where a_{ji}^* is the greatest element in $\{a_{ji} : j \in [s_i]\}$, $i \in [m]$, and let $\boldsymbol{a}^* = (\boldsymbol{a}_i^*)_{i\in[m]}$. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_i)_{i\in[m]}$ and assume the CK-class includes a moment independence assumption of order $\boldsymbol{a}^*$. The IDSS is then adequate if panel G_i delivers the vectors of expectations $\boldsymbol{\mu}_i(d)$, for all $i \in [m]$ and $d \in \mathcal{D}$.*

The proof of this result is given in Appendix A.1. Proposition 1 and Corollary 1 formalize the independence conditions required for an IDSS to be adequate, under the assumption of an algebraic conditional EU.

Example 7 (Example 4 continued) Equation (6) specifies the independence conditions that the panels must be ready to agree upon. Under these conditions the IDSS for food security is adequate if, calling $\mu_{ij} = \mathbb{E}(\theta_i j)$, the four panels deliver the following beliefs:

$$\begin{aligned}\boldsymbol{\mu}_4(d) &= (\mu_{04}, \mu_{14}), & \boldsymbol{\mu}_3(d) &= (\mu_{03}, \mu_{13}, \mu_{23}),\\ \boldsymbol{\mu}_2(d) &= (\mu_{02}, \mu_{12}), & \boldsymbol{\mu}_1(d) &= (\mu_{01}).\end{aligned}$$

In practice it is often the case that a conditional EU is algebraic (e.g. Madsen and Jensen, 2005, and in Examples 2 and 3). However, there are particular families of utility factorizations and statistical models that *ensure* the associated conditional EU is algebraic. We define these classes below and prove that their associated conditional EU is algebraic.

Definition 6 Let $\boldsymbol{Y}_i$ be the vector overseen by panel G_i, $i \in [m]$. A utility function over $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_m$ is called *panel separable* if it factorizes as

$$u(\boldsymbol{y}_1, \dots, \boldsymbol{y}_m, d) = \sum_{I \in \mathcal{P}_0([m])} k_I \prod_{i \in I} u_i(\boldsymbol{y}_i, d),$$

where $\mathcal{P}_0$ is the power set without the empty set and k_I is a criterion weight.

Definition 7 Under the conditions of Definition 6, a utility function over $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_m$ is called *additive panel separable* if it factorizes as

$$u(\boldsymbol{y}_1, \dots, \boldsymbol{y}_m, d) = \sum_{i \in [m]} k_i u_i(\boldsymbol{y}_i, d).$$

Notice that the utility function of our running example in food security in equation (1) is indeed (additive) panel separable.

Under the assumption of an (additive) panel separable utility, each panel can model its preferences over the variables under its jurisdiction using a marginal utility function of its choice. A large class of utilities, often used in practice, are *polynomial* (Müller and Machina, 1987). For simplicity, we here consider only the case when marginal utility functions have univariate arguments.

Definition 8 A *polynomial* utility function over y_i of degree n_i is defined as

$$u(y_i, d) = \sum_{j \in [n_i]} \rho_{ij}(d) y_i^j,$$

where the coefficients $\rho_{ij}(d) \in \mathbb{R}$ and the domain of y_i need to satisfy some constraints.[2]

An explicit derivation of the required constraints can be found in Keeney and Raiffa (1976) and Müller and Machina (1987).

The probabilistic model class we consider here is a specific *structural equation model* (SEM) (Bowen and Guo, 2011; Wall and Amemiya, 2000), where each variable is defined through a polynomial function. Henceforth we call these a *polynomial SEM*. SEMs were first introduced as a modelling approach in the social sciences (Westland, 2015) and are nowadays widely used especially in the causal literature (Pearl, 2000).

[2] For simplicity, we assume the intercept to be equal to zero since utilities are unique up to positive affine transformations.

Definition 9 Let $\boldsymbol{Y} = (Y_i)_{i\in[m]}$ be a random vector. A polynomial SEM is defined by

$$Y_i = \sum_{\boldsymbol{a}_i \in A_i} \theta_{i\boldsymbol{a}_i} \boldsymbol{Y}_{[i-1]}^{\boldsymbol{a}_i} + \varepsilon_i, \qquad\qquad i \in [m],$$

where $A_i \subset \mathbb{Z}_{\geq 0}^{i-1}$, ε_i is a random error with mean zero and variance ψ_i, $\theta_{i\boldsymbol{a}_i}$ is a parameter, $i \in [m]$, $\boldsymbol{a}_i \in A_i$, and $\boldsymbol{Y}_{[i-1]} = (Y_j)_{j\in[i-1]}$, with $[0] = \emptyset$.

The model assumed for our running example in food security and reported in equation (4) is an instance of a polynomial SEM.

An alternative formulation of a polynomial SEM in terms of distributions is

$$Y_i \mid (\boldsymbol{\theta}_i, \boldsymbol{Y}_{[i-1]}) \sim \left(\sum_{\boldsymbol{a}_i \in A_i} \theta_{i\boldsymbol{a}_i} Y_{[i-1]}^{\boldsymbol{a}_i}, \psi_i \right),$$

where $\boldsymbol{\theta}_i = (\theta_{i\boldsymbol{a}_i})_{\boldsymbol{a}_i \in A_i}$ and $i \in [m]$. These models are suitable candidates for a CK-class since their definition is qualitative in nature and requires only the specification of the relationships between the random variables together with a few selected moments.

For polynomial SEMs and panel separable utilities the following holds.

Theorem 1 *Assume panel G_i is responsible for Y_i, $i \in [m]$ and that the CK-class of an IDSS includes a panel separable utility and a polynomial SEM. Assume also that each panel agreed to model its marginal utility with a polynomial utility function. Then, under quasi independence, the IDSS is score separable.*

The proof of this theorem is given in Appendix A.2. Theorem 1 together with Lemma 1 shows that IDSSs, whose CK-class includes polynomial SEMs and panel separable utilities, can uniquely compute EU scores from the individual judgements of the panels. By construction, the quasi independence condition of Theorem 1 actually corresponds to a moment independence. The order of such independence depends on the polynomial form of both the SEM and the utility function. In Section 7 we identify the order of the moment independence condition required for adequacy in a subclass of polynomial SEMs.

7 Adequacy conditions for Bayesian networks

The subclass of polynomial SEMs we study next consists of BN models where each vertex is defined by a linear regression over its parents. For this model class we are able to deduce the exact moment independence required for adequacy.

Definition 10 A BN over a directed acyclic graph $\mathcal{G}$ with vertex set $V(\mathcal{G}) = \{i : i \in [m]\}$ and edge set $E(\mathcal{G})$ is a *linear SEM* if each variable Y_i is defined as

$$Y_i = \theta_{0i} + \sum_{j \in \Pi_i} \theta_{ji} Y_j + \varepsilon_i, \tag{8}$$

where Π_i is the parent set of i in $\mathcal{G}$, ε_i is a random error with mean zero and variance ψ_i and $\theta_{0i}, \theta_{ji} \in \mathbb{R}$.

Although such models are often multivariate Gaussian, in general this does not need to be the case. Notice that the model assumed for our running example in food security and reported in equation (4) is an instance of a linear SEM with respect to the BN in Figure 1.

As Sullivant et al (2010), we consider regression parameters as indeterminates in a polynomial function. We associate these to edges and vertices of the underlying graph. For $i \in [m]$, let $\theta'_{0i} = \theta_{0i} + \varepsilon_i$ be the indeterminate associated to the vertex i, whilst θ_{ij} is associated to the edge $(i,j) \in E(\mathcal{G})$.[3] Define $\mathbb{P}_i$ to be the set of rooted paths in $\mathcal{G}$ ending in Y_i. A *rooted path* of length $n+1$ from i_1 to j_n is a sequence comprising of a vertex in $V(\mathcal{G})$ and n distinct edges in $E(\mathcal{G})$ is such that $(i_1, (i_1, j_1), \dots, (i_k, j_k), (i_{k+1}, j_{k+1}), \dots, (i_n, j_n))$, where $j_k = i_{k+1}$, $k \in [n-1]$, $i_k, j_k \in [m]$. For every element $P \in \mathbb{P}_i$ we define $\boldsymbol{\theta}_P$ as

$$\boldsymbol{\theta}_P = \prod_{i \in P} \theta'_{0i} \prod_{(i,j) \in P} \theta_{ij},$$

and, as Sullivant et al (2010), we call $\boldsymbol{\theta}_P$ the *path monomial*.

Example 8 Consider the BN in Figure 1 associated to the food security example. For instance, the set $\mathbb{P}_3$ is equal to

$$\{(3), (2, (2,3)), (1, (1,3)), (1, (1,2), (2,3))\},$$

and θ'_{03}, $\theta'_{02}\theta_{23}$, $\theta'_{01}\theta_{13}$ and $\theta'_{01}\theta_{12}\theta_{23}$ are the corresponding path monomials.

We call *algebraic substitution* the process of substituting the linear regression expression of a random variable of the BN, as in equation (8), into the one of the child variable. An example illustrates this process.

Example 9 For the food security BN in Figure 1, a linear SEM is given in equation (4). An algebraic substitution of the variables in the definition of Y_3 entails

$$\begin{aligned} Y_3 &= \theta_{03} + \theta_{13}(\theta_{01} + \varepsilon_1) + \theta_{23}(\theta_{02} + \theta_{12}Y_1 + \varepsilon_2) + \varepsilon_3 \\ &= \theta'_{03} + \theta_{13}\theta'_{01} + \theta_{23}\theta'_{02} + \theta_{23}\theta_{12}Y_1. \end{aligned}$$

The additional algebraic substitution of Y_1 gives

$$Y_3 = \theta'_{03} + \theta_{13}\theta'_{01} + \theta_{23}\theta'_{02} + \theta_{23}\theta_{12}\theta'_{01}. \tag{9}$$

It is of special interest that after this substitution Y_3 is now uniquely defined in equation (9) in terms of path monomials. Proposition 2 formalizes that this occurs for any variable of a BN defined as a linear SEM and links algebraic substitutions to conditional expectation operators.

[3] We think of θ'_{0i} as a parameter although this consists of the sum of a parameter θ_{01} and an error ε_i. Note however that from a Bayesian viewpoint these are both random variables.

Proposition 2 *For a linear SEM over a directed acyclic graph $\mathcal{G}$, through algebraic substitutions each variable Y_i, $i \in [m]$, can be written as*

$$Y_i = \sum_{P \in \mathbb{P}_i} \boldsymbol{\theta}_P, \tag{10}$$

and letting $\boldsymbol{\theta}_i = (\theta'_{0i}, \theta_{ji})_{j \in \Pi_i}$ and $\boldsymbol{\theta} = (\boldsymbol{\theta}_i)_{i \in [m]}$, $i \in [m]$, we then have that

$$\mathbb{E}(Y_i \mid \boldsymbol{\theta}, d) = \sum_{P \in \mathbb{P}_i} \boldsymbol{\theta}_P. \tag{11}$$

The proof of this result is given in Appendix A.3.

7.1 Additive factorizations.

By Proposition 2, the conditional EU of polynomial additive panel separable utilities can be written as a polynomial function of a set of monomials readable into the structure of the graph.

Lemma 2 *Consider a linear SEM over a directed acyclic graph $\mathcal{G}$. Assume that $u(\boldsymbol{y})$ can be written as*

$$u_i(\boldsymbol{y}) = \sum_{i \in [m]} k_i u_i(y_i).$$

and that u_i is a polynomial utility function of degree n_i. Then the conditional EU is algebraic and can be written as

$$\bar{u}(d \mid \boldsymbol{\theta}) = \sum_{i \in [m]} k_i \sum_{j \in [n_i]} \rho_{ij}(d) \sum_{|\boldsymbol{a}_i| = j} \binom{j}{\boldsymbol{a}_i} \boldsymbol{\theta}_{\mathbb{P}_i}^{\boldsymbol{a}_i}, \tag{12}$$

where $\boldsymbol{a}_i = (a_{ij})_{j \in [\#\mathbb{P}_i]} \in \mathbb{Z}_{\geq 0}^{\#\mathbb{P}_i}$, $\boldsymbol{\theta}_{\mathbb{P}_i} = \prod_{P \in \mathbb{P}_i} \boldsymbol{\theta}_P$, $\binom{j}{\boldsymbol{a}_i}$ is a multinomial coefficient, $\#\mathbb{P}_i$ is the number of elements in $\mathbb{P}_i$ and $|\boldsymbol{a}_i| = \sum_{j \in \#\mathbb{P}_i} a_{ij}$.

This result follows by noting, from Equation (11), that the conditional EU equals

$$\mathbb{E}(\bar{u}(d \mid \boldsymbol{\theta})) = \sum_{i \in [m]} k_i \sum_{j \in [n_i]} \rho_{ij}(d) \Big(\sum_{P \in \mathbb{P}_i} \boldsymbol{\theta}_P \Big)^j,$$

and then applying the Multinomial Theorem (Cox et al, 2007).

Equation (12) is an instance of the computation of the moments of a decomposable function as studied in Cowell et al (1999) and Nilsson (2001). In Lemma 2 we explicitly deduce the required monomials and their degree and in Section 7.2 we generalise these results to multilinear functions.

Lemma 2 has an appealing intuitive graphical interpretation which is particularly useful for the computation of the EU's monomials. The j-th non central moment of any Y_i can be written as the sum of the monomials $\boldsymbol{\theta}_{\mathbb{P}_i}$ with degree j. By the properties of multinomial coefficients, this sum can be thought of as the sum over the set of unordered j-tuples of rooted paths ending in Y_i. Let $\mathbb{P}_i^j$ be the set of unordered j-tuples from $\mathbb{P}_i$. For a $P \in \mathbb{P}_i^j$, the multinomial coefficient in equation (12) counts the distinct permutations of the elements of P, denoted as n_{P_i}. We then have that,

$$\sum_{|\boldsymbol{a}_i|=j} \binom{j}{\boldsymbol{a}_i} \boldsymbol{\theta}_{\mathbb{P}_i}^{\boldsymbol{a}_i} = \sum_{P\in\mathbb{P}_i^j} n_{P_i} \prod_{p\in P} \boldsymbol{\theta}_p. \tag{13}$$

Equation (13) shows an intuitive graphical interpretation of equation (12), as illustrated in the following example.

Example 10 For the vertex 4 in the BN of Figure 1 the set $\mathbb{P}_4$ is equal to $\{(4),(1,(1,4))\}$. From the left hand side of equation (13), Y_4^2 can be written as

$$\theta_{04}'^2 + \theta_{01}'^2\theta_{14}^2 + 2\theta_{01}'\theta_{14}\theta_{04}'. \tag{14}$$

This polynomial can be also deduced by simply looking at the graph. Note that

$$\mathbb{P}_4^2 = \Big\{\big((4),(4)\big),\big((1,(1,4)),(1,(1,4))\big),\big((4),(1,(1,4))\big)\Big\}.$$

The first and second monomial in equation (14) correspond to the first and second element of $\mathbb{P}_4^2$ respectively, whilst the last elements of this set, having two distinct permutations, is associated to the third monomial in equation (14).

From Lemma 2 we can deduce the independences needed for adequacy in IDSSs whose CK class includes a BN defined as a linear SEM. Note that $\boldsymbol{\theta}_{\mathbb{P}_i}$, defined as $\prod_{P\in\mathbb{P}_i}\boldsymbol{\theta}_P$, might include multiple times the same parameter, θ say, if θ is associated to a vertex/edge appearing in different paths ending in Y_i. We let $\boldsymbol{\theta}_{\mathcal{G}_i}$ be the simplified version of $\boldsymbol{\theta}_{\mathbb{P}_i}$ where each parameter appears only once and $\boldsymbol{\theta}_{\mathcal{G}_i}^{\boldsymbol{c}_i}$ is the simplified version of $\boldsymbol{\theta}_{\mathbb{P}_i}^{\boldsymbol{a}_i}$ where each element of $\boldsymbol{c}_i$ equals the sum of the a_{ij} associated to the same parameter. Let r_i be the number of distinct parameters in $\boldsymbol{\theta}_{\mathbb{P}_i}$.

Theorem 2 *Suppose the CK-class of an IDSS includes a linear SEM over a directed acyclic graph $\mathcal{G}$, where panel G_i oversees Y_i, $i \in [m]$, and an additive panel separable utility function. Suppose panel G_i agreed to use a polynomial utility function of degree n_i, $i \in [m]$. If $\boldsymbol{\theta}_{\mathcal{G}_i}$ satisfies moment independence of order $\boldsymbol{c}_i$ for every $\boldsymbol{c}_i \in \mathbb{Z}_{\geq 0}^{r_i}$ such that $|\boldsymbol{c}_i| = n_i$ and $i \in [m]$, then the IDSS is adequate.*

The proof of this result is given in Appendix A.4. Theorem 2 gives the specific moment independences necessary for the IDSS's adequacy. By requesting the collective to agree on these independences, the IDSS can then quickly produce a unique EU score for each policy. Panels are informed on the summaries they need to deliver to the IDSS since these are the only quantities of which the EU is a function. An illustration of this result has already been given in Examples 3, 5, 7 for our food security running example.

7.2 Multilinear factorizations.

By approaching the combination of outputs in BN models from an algebraic viewpoint, we are able to generalize in a straightforward manner the results in Section 7.1 about additive/decomposable factorizations so that they apply to multilinear functions. Let $\#\mathbb{P}_i = m_i$, i.e. there are m_i rooted paths ending in Y_i. Let $\boldsymbol{l}_i = (l_{ij})_{j\in[m_i]} \in \mathbb{Z}_{\geq 0}^{m_i}$ be the vector listing the lengths of such paths and $\boldsymbol{l} = (\boldsymbol{l}_i)_{i\in[m]}$. For a vector $\boldsymbol{a} = (a_i)_{i\in[m]} \in \mathbb{Z}^m$, we write $\boldsymbol{l} \simeq \boldsymbol{a}$ if both $|\boldsymbol{a}| = |\boldsymbol{l}|$ and, for all $i \in [m]$, $|\boldsymbol{l}_i| = a_i$.

Lemma 3 *For a linear SEM over a directed acyclic graph $\mathcal{G}$, suppose the utility function $u(\boldsymbol{y}, d)$ can be written*

$$u(\boldsymbol{y}, d) = \sum_{I\in\mathcal{P}_0([m])} k_I \prod_{i\in I} u_i(y_i, d).$$

Now suppose u_i is a polynomial utility function of degree n_i, $\boldsymbol{n} = (n_i)_{i\in[m]}$, $i \in [m]$ and $\mathbf{0}$ is a vector of dimension m with only zero entries. The conditional EU is then algebraic and can be written as

$$\bar{u}(d \mid \boldsymbol{\theta}) = \sum_{\mathbf{0}<_{lex}\boldsymbol{a}\leq_{lex}\boldsymbol{n}} c_{\boldsymbol{a}}(d) \sum_{\boldsymbol{l}\simeq\boldsymbol{a}} \binom{|\boldsymbol{a}|}{\boldsymbol{l}} \boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{l}}, \tag{15}$$

where $c_{\boldsymbol{a}}(d) = k_J \prod_{j\in J} \rho_{ja_j}(d)$, $J = \{j \in [m] : a_j \neq 0\}$, and $\boldsymbol{\theta}_{\mathbb{P}} = \prod_{i\in[m]} \boldsymbol{\theta}_{\mathbb{P}_i}$.

The proof of Lemma 3 is given in Appendix A.5 Lemma 3 makes a significant generalization to the theory of the computation of moments in decomposable/additive functions of Cowell et al (1999) and Nilsson (2001) extending these well-known formulae so that they apply in the much wider context of multilinear functions of BNs defined as linear SEMs. It is interesting to note that Lemma 3 is connected to the propagation algorithms first developed in Lauritzen (1992) to compute the first two moments of certain chain graphs. Here, focusing on a specific class of continuous BN models, we are able to explicitly compute, through algebraic substitution, not only the first two moments, but also any other higher order moment of the distribution associated with the graph.

Using again the properties of multinomial coefficients, we can relate equation (15) to the topology of the graph and its rooted paths. For an $\boldsymbol{a} \in \mathbb{Z}_{\geq 0}^m$, let $\mathbb{P}_{\boldsymbol{a}} = \times_{a_i\neq 0}\mathbb{P}_i^{a_i}$, where $\times$ denotes the Cartesian product. This set consists of the unordered $|\boldsymbol{a}|$-tuples of rooted paths, where in each tuple there are a_i paths ending at Y_i. For each element $P \in \mathbb{P}_{\boldsymbol{a}}$, let $n_P = \sum_{a_i\neq 0} n_{P_i}$. Then we have that, following the same reasoning outlined for additive factorizations,

$$\sum_{\boldsymbol{l}\simeq\boldsymbol{a}} \binom{|\boldsymbol{a}|}{\boldsymbol{l}} \boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{l}} = \sum_{P\in\mathbb{P}_{\boldsymbol{a}}} n_P \prod_{p\in P} \boldsymbol{\theta}_p.$$

Here n_P counts the total number of permutations in the sets $\mathbb{P}_i$, $i \in [m]$. This representation of non-central moments in terms of paths extends the

$((2),(2),(4),(4))$
$((1,(1,2)),(2),(4),(4))$
$((1,(1,2)),(1,(1,2)),(4),(4))$
$((2),(2),(1,(1,4)),(4))$
$((1,(1,2)),(2),(1,(1,4)),(4))$
$((1,(1,2)),(1,(1,2)),(1,(1,4)),(4))$
$((2),(2),(1,(1,4)),(1,(1,4)))$
$((1,(1,2)),(2),(1,(1,4)),(1,(1,4)))$
$((1,(1,2)),(1,(1,2)),(1,(1,4)),(1,(1,4)))$

Table 1 Tuples of dimension 4 with two paths ending in Y_2 and two more ending in Y_4 in the graph in Figure 1.

computation of the second central moment of Sullivant et al (2010) via the trek rule to generic non-central moments.

Example 11 Consider the expectation $\mathbb{E}(Y_2^2Y_4^2)$ for the variables in the BN of Figure 1. This expectation, being the associated monomial of degree 4, can be computed by looking at all distinct tuples of rooted paths of dimension four, where two paths end in Y_2 and two in Y_4. These are summarized in Table 1. The associated conditional expectation can be written as the following polynomial, where the i-th monomial corresponds to the tuple in the i-th row of Table 1:

$$\begin{aligned}\bar{u}(d \mid \boldsymbol{\theta}) = \theta_{02}'^2\theta_{04}'^2 + 2\theta_{12}\theta_{02}'\theta_{04}'^2 + \theta_{12}^2\theta_{04}'^2 + 2\theta_{02}'^2\theta_{14}\theta_{04}' + 4\theta_{12}\theta_{02}'\theta_{14}\theta_{04} \\ + 2\theta_{12}^2\theta_{14}\theta_{04}' + \theta_{02}'^2\theta_{14}^2 + 2\theta_{12}\theta_{02}'\theta_{14}^2 + \theta_{12}^2\theta_{14}^2.\end{aligned}$$

Note for example that $\theta_{12}\theta_{02}'\theta_{04}'^2$ has coefficient 2 since the paths (Y_2) and $(Y_1,(Y_1,Y_2))$ can be permuted, whilst $\theta_{12}\theta_{02}'\theta_{14}\theta_{04}$ has coefficient 4 since both pairs of paths (Y_2) and $(Y_1,(Y_1,Y_2))$ and (Y_4) and $(Y_1,(Y_1,Y_4))$ can be permuted.

Just as in the additive case, we are now able to deduce the independences required for score separability of an IDSS whose structural consensus includes a BN. We let $\boldsymbol{\theta}_{\mathcal{G}}^{\boldsymbol{b}}$ be the simplified version of $\boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{a}}$ where parameters only appear once and the exponent are appropriately summed.

Theorem 3 *Suppose that the CK-class of an IDSS includes a linear SEM over a directed acyclic graph $\mathcal{G}$, where panel G_i oversees Y_i, $i \in [m]$, and a panel separable utility. Suppose panel G_i agreed to use a polynomial utility function of degree n_i, $i \in [m]$. If, for every $\boldsymbol{b} \simeq \boldsymbol{n}$, where $\boldsymbol{n} = (n_i)_{i\in[m]} \in \mathbb{Z}_{\geq 0}^m$, $\boldsymbol{\theta}_{\mathcal{G}}$ satisfies moment independence of order $\boldsymbol{b}$, then the IDSS is score separable.*

The proof of this result is given in Appendix A.6. Theorem 3 ensures adequacy for a large class of IDSSs based on flexible multilinear utility factorizations and commonly used BNs defined as linear SEMs.

$k_1\rho_{11}\theta_{01}$,	$k_1\rho_{12}\theta_{01}^2$	$k_1\rho_{12}\psi_1$	$k_2\rho_{21}\theta_{02}$
$k_2\rho_{21}\theta_{01}\theta_{12}$	$k_2\rho_{22}\theta_{02}^2$	$k_2\rho_{22}\psi_2$	$k_2\rho_{22}\theta_{01}^2\theta_{12}^2$
$k_2\rho_{22}\psi_1\theta_{12}^2$	$2k_2\rho_{22}\theta_{01}\theta_{02}\theta_{12}$	$k_3\rho_{31}\theta_{03}$	$k_3\rho_{31}\theta_{02}\theta_{23}$
$k_3\rho_{31}\theta_{01}\theta_{12}\theta_{23}$	$k_3\rho_{31}\theta_{01}\theta_{13}$	$k_3\rho_{32}\theta_{03}^2$	$k_3\rho_{32}\psi_3$
$k_3\rho_{32}\theta_{01}^2\theta_{13}^2$	$k_3\rho_{32}\theta_{13}^2\psi_1$	$k_3\rho_{32}\theta_{02}^2\theta_{23}^2$	$2k_3\rho_{32}\theta_{03}\theta_{01}\theta_{12}\theta_{23}$
$k_3\rho_{32}\psi_2\theta_{23}^2$	$k_3\rho_{32}\psi_1\theta_{12}^2\theta_{23}^2$	$k_3\rho_{32}\theta_{01}^2\theta_{12}^2\theta_{23}^2$	$2k_3\rho_{32}\theta_{03}\theta_{02}\theta_{23}$
$2k_3\rho_{32}\theta_{03}\theta_{01}\theta_{13}$	$2k_3\rho_{32}\psi_1\theta_{12}\theta_{13}\theta_{23}$	$2k_3\rho_{32}\theta_{01}\theta_{02}\theta_{13}\theta_{23}$	$2k_3\rho_{32}\theta_{12}\theta_{13}\theta_{23}\theta_{01}^2$
$2k_3\rho_{32}\theta_{01}\theta_{02}\theta_{12}\theta_{23}^2$	$k_4\rho_{41}\theta_{04}$	$k_4\rho_{41}\theta_{01}\theta_{14}$	$k_4\rho_{42}\theta_{04}^2$
$k_4\rho_{42}\psi_4$	$k_4\rho_{42}\theta_{01}^2\theta_{14}^2$	$k_4\rho_{42}\psi_1\theta_{14}^2$	$2k_4\rho_{42}\theta_{01}\theta_{04}\theta_{14}$

Table 2 Monomials of the conditional EU for the utility class $\mathbb{U}_1$.

8 An application in household food security

The running example of food security we have used to illustrate our methodology so far is really simple and based on a simple additive factorization of linear utility function. To illustrate the application of our results in a much more credible, real-world example, we consider the same food security network reported in Figure 1, but now under a more general scenario. Suppose the variables are each under the jurisdiction of a different panel of experts. Jointly they reach a consensus to:

- investigate the effectiveness of an increase (d_0), decrease (d_1) or not a change (d_2) of the number of pupils eligible for free school meals, with $\mathbb{D} = \{d_0, d_1, d_2\}$ (decision consensus);
- model the conditional dependences between the four random variables deemed relevant with the BN reported in Figure 1 (structural consensus);
- consider two utility classes of utility factorizations - the first with preferentially independent attributes (class $\mathbb{U}_1$) as in equation (1), the second enjoying a multilinear utility factorization (class $\mathbb{U}_2$) defined as

$$u(d, \boldsymbol{y}) = \sum_{I \in \mathcal{P}_0([4])} k_I \prod_{i \in I} u_i(y_i, d).$$

These agreements give the CK-class for this application. Next suppose that each panel decides to model the variable under its jurisdiction via a linear SEM as specified in equation (4) and to model its marginal utility with a polynomial utility function of degree two, i.e. $u_i(y_i, d) = \rho_{i1}(d)y_i + \rho_{i2}(d)y_i^2$, $i \in [4]$.

The two questions that we next address are the following: what independences do the panels need to be prepared to make for the IDSS to be adequate? What summaries do they have to deliver? The answer depends on the class of utility functions chosen. We thus first focus on the simpler class $\mathbb{U}_1$ of preferentially independent attributes.

Through the process of algebraic substitutions, as formalized in Lemma 2, the conditional EU for the utility class $\mathbb{U}_1$ can be computed as the sum of the monomials reported in Table 2 where we left the dependence on the policy $d \in \mathbb{D}$ implicit. Given this list of monomials, it is then straightforward the identify the independences required by the IDSS to be able to compute the

	$\mathbb{E}(\theta_{01})$	$\mathbb{E}(\psi_1)$	$\mathbb{E}(\theta_{04})$	$\mathbb{E}(\psi_4)$	$\mathbb{E}(\theta_{02})$	$\mathbb{E}(\psi_2)$	$\mathbb{E}(\theta_{12})$
d_0	1.5	5	30	8	5	40	7
d_1	-2	4	-5	5	-6	20	2
d_2	-0.5	3	10	4	3	15	7

Table 3 Probabilistic panel specifications that depend on the policy taken for the food security application.

$\mathbb{E}(\theta_{03}) = 5,$	$\mathbb{E}(\theta_{13}) = 17,$	$\mathbb{E}(\theta_{23}) = 10,$	$\mathbb{E}(\theta_{14}) = 10,$	$\mathbb{E}(\psi_3) = 20,$
$\mathbb{V}(\theta_{01}) = 1,$	$\mathbb{V}(\theta_{02}) = 1,$	$\mathbb{V}(\theta_{03}) = 1,$	$\mathbb{V}(\theta_{04}) = 1,$	$\mathbb{V}(\theta_{12}) = 1,$
$\mathbb{V}(\theta_{12}) = 1,$	$\mathbb{V}(\theta_{13}) = 3,$	$\mathbb{V}(\theta_{14}) = 2,$	$\mathbb{V}(\theta_{23}) = 2,$	

Table 4 Probabilistic panel specifications independent of the policy taken for the food security application.

$k_1 = 0.25,$	$k_2 = 0.25,$	$k_3 = 0.25,$	$k_4 = 0.25,$
$\rho_{11} = -2,$	$\rho_{12} = 1,$	$\rho_{21} = 2,$	$\rho_{22} = 10,$
$\rho_{31} = 8,$	$\rho_{32} = 0.5,$	$\rho_{41} = 3,$	$\rho_{42} = -5,$

Table 5 Criterion weights and terms in the utility functions for the food security application.

EU of any available policy as a function of beliefs individually delivered by panels. Specifically, the moment independences summarized in Table 6 need to hold. Assuming these, then each panel can deliver independently the required beliefs to derive appropriate EU scores uniquely.

Assume the panels were capable of deliver the following beliefs: parameter estimates that are directly affected by the policy taken (reported in Table 3), parameter estimates which are independent of the policy taken (reported in Table 4) and the criterion weights as well as the parameters of the utility functions (reported in Table 5)[4]. Given these beliefs independently delivered by each panels then the IDSS, under the separation conditions reported in Table 2, would recommend that the number of pupils eligible for free school meals is increased since the EU of this policy equals 1.87, whilst for d_1 and d_2 this is 0.51 and 0.62 respectively.

We next consider the case where the CK-class includes the second class $\mathbb{U}_2$ of multilinear utilities. Whilst for preferentially independent attributes the conditional EU has 36 monomials (in Table 2), in this case the conditional EU can be shown to have 3869 monomials. In this case the computer algebra software Mathematica instantaneously gives us the conditional EU polynomial using the simple code reported in Appendix B. The output conditional EU can then be used to identify the required moment independences and panels' beliefs. For instance, the parameter θ_{01} has degree up to 8 in the polynomial conditional EU when $\mathbb{U}_2$ is used, whilst for $\mathbb{U}_1$ its maximum degree was 2. Using this more general class of utilities and the criterion weights' specifications

[4] Notice that these values are then normalized to give utility functions between 0 and 1.

$\mathbb{E}(\theta_{01}\theta_{12}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})$	$\mathbb{E}(\theta_{01}^2\theta_{12}^2) = \mathbb{E}(\theta_{01}^2)\mathbb{E}(\theta_{12}^2)$
$\mathbb{E}(\psi_1\theta_{12}^2) = \mathbb{E}(\psi_1)\mathbb{E}(\theta_{12}^2)$	$\mathbb{E}(\theta_{01}\theta_{02}\theta_{12}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{02}\theta_{12})$
$\mathbb{E}(\theta_{02}\theta_{23}) = \mathbb{E}(\theta_{02})\mathbb{E}(\theta_{23})$	$\mathbb{E}(\theta_{01}\theta_{12}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{23})$
$\mathbb{E}(\theta_{01}\theta_{13}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{13})$	$\mathbb{E}(\theta_{01}^2\theta_{13}^2) = \mathbb{E}(\theta_{01}^2)\mathbb{E}(\theta_{13}^2)$
$\mathbb{E}(\theta_{13}^2\psi_1) = \mathbb{E}(\theta_{13}^2)\mathbb{E}(\psi_1)$	$\mathbb{E}(\theta_{02}^2\theta_{23}^2) = \mathbb{E}(\theta_{02}^2)\mathbb{E}(\theta_{23}^2)$
$\mathbb{E}(\theta_{01}\theta_{12}\theta_{03}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{03}\theta_{23})$	$\mathbb{E}(\theta_{23}^2\psi_2) = \mathbb{E}(\theta_{23}^2)\mathbb{E}(\psi_2)$
$\mathbb{E}(\psi_1\theta_{12}^2\theta_{23}^2) = \mathbb{E}(\psi_1)\mathbb{E}(\theta_{12}^2)\mathbb{E}(\theta_{23}^2)$	$\mathbb{E}(\theta_{01}^2\theta_{12}^2\theta_{23}^2) = \mathbb{E}(\theta_{01}^2)\mathbb{E}(\theta_{12}^2)\mathbb{E}(\theta_{23}^2)$
$\mathbb{E}(\theta_{02}\theta_{03}\theta_{23}) = \mathbb{E}(\theta_{02})\mathbb{E}(\theta_{03}\theta_{23})$	$\mathbb{E}(\theta_{01}\theta_{03}\theta_{13}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{03}\theta_{13})$
$\mathbb{E}(\psi_1\theta_{12}\theta_{13}\theta_{23}) = \mathbb{E}(\psi_1)\mathbb{E}(\theta_{12})\mathbb{E}(\theta_{13}\theta_{23})$	$\mathbb{E}(\theta_{01}\theta_{02}\theta_{13}\theta_{23}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{02})\mathbb{E}(\theta_{13}\theta_{23})$
$\mathbb{E}(\theta_{01}\theta_{02}\theta_{12}\theta_{23}^2) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{02}\theta_{12})\mathbb{E}(\theta_{23}^2)$	$\mathbb{E}(\theta_{12}\theta_{13}\theta_{23}\theta_{01}^2) = \mathbb{E}(\theta_{12})\mathbb{E}(\theta_{13}\theta_{23})\mathbb{E}(\theta_{01}^2)$
$\mathbb{E}(\theta_{01}\theta_{14}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{14})$	$\mathbb{E}(\theta_{01}^2\theta_{14}^2) = \mathbb{E}(\theta_{01}^2)\mathbb{E}(\theta_{14}^2)$
$\mathbb{E}(\psi_1\theta_{14}^2) = \mathbb{E}(\psi_1)\mathbb{E}(\theta_{14}^2)$	$\mathbb{E}(\theta_{01}\theta_{04}\theta_{14}) = \mathbb{E}(\theta_{01})\mathbb{E}(\theta_{04}\theta_{14})$

Table 6 Moment independences required by the IDSS for adequacy.

$k_1 = 0.15,$	$k_2 = 0.15,$	$k_3 = 0.15,$	$k_4 = 0.15,$	$k_{12} = 0.05,$
$k_{13} = 0.05,$	$k_{14} = 0.05,$	$k_{23} = 0.05,$	$k_{24} = 0.05,$	$k_{34} = 0.05,$
$k_{123} = 0.02,$	$k_{124} = 0.02,$	$k_{134} = 0.02,$	$k_{234} = 0.02,$	$k_{1234} = 0.02,$

Table 7 Criterion weights' specification for the food security example under the utility class $\mathbb{U}_2$.

reported in Table 7[5], the IDSS would again recommend that the number of pupils eligible for free school meals is increased since the EU of this policy equals 0.97, whilst for d_1 and d_2 this is 0.16 and 0.37 respectively.

9 Discussion

The framework of IDSSs is capable of supporting decision making in situations where judgements come from different panels of experts having jurisdiction over different aspects of the system. In this paper we have relaxed many of the assumptions guaranteeing coherence in this type of systems (Smith et al, 2015) by exploiting the polynomial structure of certain statistical models and utility functions and illustrated their usefulness in a practical application.

In the particular context where the structural consensus includes a BN model, the process of algebraic substitution has proven fundamental in identifying the required summaries and independence relations. We have encouraging results, to be reported in future work, towards a generalization of such recursions in dynamic models, as the multiregression dynamic model (Queen and Smith, 1993), where expressions for the moments' forecasts can be deduced in closed form. Furthermore, when each vertex of the BN is no longer a random variable but a random vector (for example when a variable is measured at different geographic location), the theory of tensors (McCullagh, 1987) can be employed to concisely report the associated EU expressions. We plan to develop such a methodology in future work.

[5] In the multilinear case higher moments are required. Here we assume that these can be computed from the first two moments using the recursions of normal distributions.

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A Proofs

A.1 Proof of Corollary 1

Adequacy is guaranteed if the EU function can be written in terms of $\mu_{ji}(d)$ and $k_{\boldsymbol{b}}$, $i \in [m]$, $j \in [s_i]$ and $d \in \mathcal{D}$. Note that

$$\begin{aligned}\bar{u}(d) &= \mathbb{E}\left(\bar{u}(d|\boldsymbol{\theta})\right)\\ &= \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \mathbb{E}\Big(\prod_{i\in[m]}\prod_{j\in[s_i]^0} \lambda_{ji}(\boldsymbol{\theta}_i, d)^{b_{j,i}}\Big)\\ &= \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \mathbb{E}\Big(\prod_{i\in[m]}\prod_{j\in[s_i]^0} \boldsymbol{\theta}_i^{\boldsymbol{a}_{ji}}\Big).\end{aligned}$$

The argument of this expectation is a monomial of multi-degree lower or equal to $\boldsymbol{a}^*$. Moment independence then implies that $\bar{u}(d) = \sum_{\boldsymbol{b}\in B} k_{\boldsymbol{b}} \prod_{i\in[m]}\prod_{j\in[s_i]^0} \mu_{ji}(d)$, and the result follows.

A.2 Proof of Theorem 1

Fix a policy $d \in \mathbb{D}$ and suppress this dependence. Under the assumptions of the theorem, the utility function can be written as

$$u(\boldsymbol{y}) = \sum_{I\in\mathcal{P}_0([m])} k_I \sum_{i\in I}\left(\sum_{j\in[n_i]} \rho_{ij} y_i^j\right). \tag{16}$$

Note also that we can rewrite (16) as

$$u(\boldsymbol{y}) = \hat{u}(\boldsymbol{y}_{[m-1]}) + \hat{u}(y_m),$$

where

$$\begin{aligned}\hat{u}(\boldsymbol{y}_{[m-1]}) &= \sum_{I\in\mathcal{P}_0([m-1])} k_I \prod_{i\in I}\Big(\sum_{j\in[n_i]} \rho_{ij} y_i^j\Big),\\ \hat{u}(y_m) &= \sum_{I\in\mathcal{P}_0^m([m])} k_I \prod_{i\in I}\Big(\sum_{j\in[n_i]} \rho_{ij} y_i^j\Big),\end{aligned} \tag{17}$$

and $\mathcal{P}_0^m([m]) = \mathcal{P}_0([m]) \cap \{m\}$. Calling $\boldsymbol{\theta}$ the overall parameter vector of the IDSS, the conditional EU function can be written applying sequentially the tower rule of expectation as

$$\mathbb{E}(u(\boldsymbol{Y}) \mid \boldsymbol{\theta}) = \mathbb{E}_{Y_1|\boldsymbol{\theta}}\Big(\cdots \mathbb{E}_{Y_{m-1}|\boldsymbol{Y}_{[m-2]},\boldsymbol{\theta}}\big(\hat{u}(\boldsymbol{y}_{[m-1]}) + \mathbb{E}_{Y_m|\boldsymbol{Y}_{[m-1]},\boldsymbol{\theta}}(\hat{u}(\boldsymbol{y}_m))\big)\Big). \tag{18}$$

From equation (17), the definition of a polynomial SEM and observing that the power of a polynomial is still a polynomial function in the same arguments, it follows that

$$E_{Y_m|\boldsymbol{Y}_{[m-1]},\boldsymbol{\theta}}\left(\hat{u}(\boldsymbol{y}_m)\right) = p_m(\boldsymbol{Y}_{[m-1]}, \boldsymbol{\theta}),$$

where p_m is a generic polynomial function. Thus $\hat{u}(\boldsymbol{Y}_{[m-1]}) + \mathbb{E}_{Y_m|\boldsymbol{Y}_{[m-1]},\boldsymbol{\theta}}\left(\hat{u}(\boldsymbol{y}_m)\right)$ is also a polynomial function in the same arguments. Following the same reasoning, we have that

$$\mathbb{E}_{Y_{m-1}|\boldsymbol{Y}_{[m-2]},\boldsymbol{\theta}}\left(\hat{u}(\boldsymbol{y}_{[m-1]}) + \mathbb{E}_{Y_m|\boldsymbol{Y}_{[m-1]},\boldsymbol{\theta}}\left(\hat{u}(\boldsymbol{y}_m)\right)\right) = p_{m-1}(\boldsymbol{Y}_{[m-2]}, \boldsymbol{\theta}),$$

where p_{m-1} is a generic polynomial function. Therefore the same procedure can be applied to all the expectations in (18). So $\mathbb{E}(u(\boldsymbol{Y}) \mid \boldsymbol{\theta}) = p_1(\boldsymbol{\theta})$, where p_1 is a generic polynomial function. This defines by construction an algebraic conditional EU, where the functions λ_{ij} are monomials. Quasi independence and Lemma 1 then guarantee score separability holds.

A.3 Proof of Proposition 2

We prove equation (10) via induction over the indices of the variables. Let Y_1 be a root of $\mathcal{G}$. Thus $Y_1 = \theta'_{01}$, where θ'_{01} is the monomial associated to the only rooted path ending in Y_1, namely (Y_1). Assume the result is true for Y_{n-1} and consider Y_n. By the inductive hypothesis we have that, if $i < j$ whenever $i \in \Pi_j$,

$$Y_n = \theta'_{0n} + \sum_{i\in\Pi_n} \theta_{in} Y_i = \theta'_{0n} + \sum_{i\in\Pi_n} \theta_{in} \sum_{P\in\mathbb{P}_i} \boldsymbol{\theta}_P. \tag{19}$$

Note that every rooted path ending in Y_n is either (Y_n) or consists of a rooted path ending in Y_i, $i \in \Pi_n$, together with the edge (Y_i, Y_n). From this observation the result then follows by rearranging the terms in equation (19). Equation (11) can be proven via the same inductive process noting that $\mathbb{E}(Y_1 \mid \boldsymbol{\theta}, d) = \theta'_{01}$ and $\mathbb{E}(Y_n \mid \boldsymbol{\theta}, d) = \theta'_{0n} + \sum_{i\in\Pi_n} \theta_{in}\mathbb{E}(Y_i \mid \boldsymbol{\theta}, d)$.

A.4 Proof of Theorem 2

Under the assumptions of the theorem, the conditional EU function can be written as in equation (12). From the linearity of the expectation operator we have that

$$\begin{aligned}\mathbb{E}(\bar{u}(d \mid \boldsymbol{\theta})) &= \sum_{i\in[m],j\in[n_i]} k_i\rho_{ij}(d) \sum_{|\boldsymbol{a}_i|=j} \binom{j}{\boldsymbol{a}_i} \mathbb{E}\left(\boldsymbol{\theta}_{\mathbb{P}_i}^{\boldsymbol{a}_i}\right) \\ &= \sum_{i\in[m],j\in[n_i]} k_i(\rho_{ij}(d) \sum_{|\boldsymbol{c}_i|=j} \binom{j}{\boldsymbol{c}_i} \mathbb{E}\left(\boldsymbol{\theta}_{\mathcal{G}_i}^{\boldsymbol{c}_i}\right).\end{aligned}$$

Applying moment independence and letting V_i and E_i be the sets of distinct vertices and edges, respectively, for all the elements $P \in \mathbb{P}_i$, we have that

$$\mathbb{E}(\bar{u}(d \mid \boldsymbol{\theta})) = \sum_{\substack{i\in[m],j\in[n_i],\\ |\boldsymbol{c}_i|=j}} k_i\rho_{ij}(d)\binom{j}{\boldsymbol{c}_i} \prod_{l\in V_i}\mathbb{E}\left(\theta_{0l}^{\prime c_{il}}\theta_{lCh_l}^{c_iCh_l}\right) \prod_{(j,k)\in E_i\setminus(l,Ch_l)} \mathbb{E}\left(\theta_{jk}^{c_{ik}}\right),$$

where c_{ik} is the element of $\boldsymbol{c}_i$ associated to θ_{jk} and Ch_l is the index of a children of the vertex l. The thesis then follows since each of these expectations is delivered by an individual panel.

A.5 Proof of Lemma 3

To prove this result we first show that under the assumptions of the lemma the utility function can be written as

$$u(\boldsymbol{y}, d) = \sum_{\boldsymbol{0}<_{lex}\boldsymbol{a}\leq_{lex}\boldsymbol{n}} c_{\boldsymbol{a}}(d)\boldsymbol{y}^{\boldsymbol{a}}, \tag{20}$$

and then prove that

$$\boldsymbol{Y}^{\boldsymbol{a}} = \sum_{\boldsymbol{l}\simeq\boldsymbol{a}} \binom{|\boldsymbol{a}|}{\boldsymbol{l}}\boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{l}}. \tag{21}$$

The lemma then follows by substituting into equation (20) for $\boldsymbol{y}^{\boldsymbol{a}}$ given in equation (21).

Fix a policy $d \in \mathbb{D}$ and suppress this dependence. We prove equation (20) via induction over the number of vertices of the BN. If the BN has only one vertex then

$$u(\boldsymbol{y}) = k_1 \sum_{i\in n_1} \rho_{1i} y_1^i.$$

This can be seen as an instance of equation (20). Assume the result holds for a network with $n-1$ vertices. A multilinear utility factorisation can be rewritten as

$$u(\boldsymbol{y}) = \sum_{I\in\mathcal{P}_0([n-1])} k_I \prod_{i\in I} u_i(y_i) + \sum_{I\in\mathcal{P}_0^n([n])} k_I \prod_{i\in I\setminus\{n\}} u_i(y_i)u_n(y_n) + k_n u_n(y_n). \quad (22)$$

The first term on the rhs of (22) is by inductive hypothesis equal to the sum of all the possible monomial of degree $\boldsymbol{a} = (a_1, \dots, a_{n-1}, 0)$ where $0 \le a_i \le n_i$, $i \in [n]$. The other terms only include monomials such that the exponent of y_n is not zero. Letting $\boldsymbol{n}_{n-1} = (n_i)_{i\in[n-1]}$, $\boldsymbol{y}_{[n-1]} = \prod_{i\in[n-1]} y_i$ and $u' = \sum_{I\in\mathcal{P}_0^n([n])} k_I \prod_{i\in I\setminus\{n\}} u_i(y_i)u_n(y_n) + k_n u_n(y_n)$, we now have that

$$\begin{aligned} u' &= \sum_{\boldsymbol{0}<_{lex}\boldsymbol{a}\le_{lex}\boldsymbol{n}_{n-1}} c_{\boldsymbol{a}} \boldsymbol{y}_{[n-1]}^{\boldsymbol{a}} \left(\sum_{i\in[n_n]} \rho_{ni} y_n^i \right) + k_n u_n(y_n) \\ &= \sum_{\substack{\boldsymbol{0}<_{lex}\boldsymbol{a}\le_{lex}\boldsymbol{n}_{n-1} \\ i\in[n_n]}} c_{\boldsymbol{a}} \rho_{ni} \boldsymbol{y}_{[n-1]}^{\boldsymbol{a}} y_n^i + k_n u_n(y_n) = \sum_{\substack{\boldsymbol{0}'<_{lex}\boldsymbol{a}\le_{lex}\boldsymbol{n}_n \\ a_n\neq 0}} c_{\boldsymbol{a}} \boldsymbol{y}_{[n]}^{\boldsymbol{a}}. \end{aligned} \quad (23)$$

Therefore, equation (20) follows from equations (22) and (23). To prove equation (21) note that the monomial $\boldsymbol{Y}^{\boldsymbol{a}}$ can be written as

$$\boldsymbol{Y}^{\boldsymbol{\alpha}} = \prod_{i\in[m]} Y_i^{a_i} = \prod_{i\in[m]} \left(\sum_{|\boldsymbol{l}_i|=a_i} \binom{a_i}{\boldsymbol{l}_i} \boldsymbol{\theta}_{\mathbb{P}_i}^{\boldsymbol{l}_i} \right) = \sum_{\boldsymbol{l}\simeq\boldsymbol{a}} \boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{l}} \prod_{i\in[m]} \binom{a_i}{\boldsymbol{l}_i}.$$

Equation (21) then follows by noting that

$$\prod_{i\in[m]} \binom{a_i}{\boldsymbol{l}_i} = \frac{\prod_{i\in[m]} a_i!}{\prod_{i\in[m]} \prod_{j\in[n_i]} l_{ij}!} = \binom{|\boldsymbol{a}|}{\boldsymbol{l}}.$$

A.6 Proof of Theorem 3

Under the conditions of the theorem, the conditional EU function can be written as in (15). The linearity of the expectation operator than implies that

$$\mathbb{E}(\bar{u}(d \mid \boldsymbol{\theta})) = \sum_{\substack{\boldsymbol{0}<_{lex}\boldsymbol{a}\le_{lex}\boldsymbol{n} \\ \boldsymbol{l}\simeq\boldsymbol{a}}} c_{\boldsymbol{a}} \binom{|\boldsymbol{a}|}{\boldsymbol{l}} \mathbb{E}\left(\boldsymbol{\theta}_{\mathbb{P}}^{\boldsymbol{l}}\right) = \sum_{\substack{\boldsymbol{0}<_{lex}\boldsymbol{b}\le_{lex}\boldsymbol{n} \\ \boldsymbol{l}\simeq\boldsymbol{b}}} c_{\boldsymbol{b}} \binom{|\boldsymbol{b}|}{\boldsymbol{l}} \mathbb{E}\left(\boldsymbol{\theta}_{\mathcal{G}}^{\boldsymbol{l}}\right).$$

Applying moment independence and letting V_{tot} and E_{tot} be the sets of distinct vertices and edges, respectively, for all the elements $P \in \mathbb{P} = \cup_{i\in[m]}\mathbb{P}_i$, we then have that for any $\boldsymbol{l} \simeq \boldsymbol{b}$

$$\mathbb{E}\left(\boldsymbol{\theta}_{\mathcal{G}}^{\boldsymbol{l}}\right) = \prod_{t\in V_{tot}} \mathbb{E}\left(\theta_{0t}^{\prime l_{it}} \theta_{tCh_t}^{l_{iCh_t}}\right) \prod_{(j,k)\in E_{tot}\setminus(t,Ch_t)} \mathbb{E}\left(\theta_{jk}^{l_{ik}}\right).$$

Score separability then follows since each of these expectations is delivered by an individual panel.

B Code for the multilinear factorization

```
y4 := t04 + t14*y1 + e4;
y3 := t03 + t13*y1 + t23*y2 + e3;
y2 := t02 + t12*y1 + e2;
```

```
y1 := t01 + e1;
u4 := c4*y4∧2 + b4*y4;
u3 := c3*y3∧2 + b3*y3;
u2 := c2*y2∧2 + b2*y2;
u1 := c1*y1∧2 + b1*y1;
u := k1*u1 + k2*u2 + k3*u3 + k4*u4 + k12*u1*u2 + k13*u1*u3 + k14*u1*u4
+ k23*u2*u3 + k24*u2*u4 + k34*u3*u4 + k123*u1*u2*u3 + k124*u1*u2*u4
+ k134*u1*u3*u4 + k234*u2*u3*u4 + k1234*u1*u2*u3*u4;
eu4 := ReplaceAll[Collect[u, e4], {e4 -> 0, e4∧2 -> psi4}];
eu3 := ReplaceAll[Collect[eu4, e3], {e3 -> 0, e3∧2 -> psi3}];
eu2 := ReplaceAll[Collect[eu3, e2], {e2 -> 0, e2∧2 -> psi2}];
eu1 := ReplaceAll[Collect[eu2, e1], {e1 -> 0, e1∧2 -> psi1}];
eu1
```